Gentzen’s sequent calculus

1935.a. Gerhard Genzen defined 1st order formulas as *sequents.*

Sequent $A_1, \ldots, A_m \vdash B_1, \ldots, B_n$

is equivalent to *1st order formula* $A_1 \wedge \ldots \wedge A_m \rightarrow B_1 \vee \ldots \vee B_n$

where $m, n \geq 0$ and $A_1, \ldots, A_m$, $B_1, \ldots, B_n$ are formulas

Sequent lh formula $A_1, \ldots, A_m$ – *antecedent,*

rh formula $B_1, \ldots, B_n$ – *succedent.*

Antecedent: $A_1, \ldots, A_m$ represents formula $A_1 \wedge \ldots \wedge A_m$

Succedent: $B_1, \ldots, B_n$ represents formula $B_1 \vee \ldots \vee B_n$.

$m = 0$, means that rh formula is unconditionally true

$n = 0$, means empty disjunct and contradiction

Language

Sequents consist of formulas constructed using : $\neg, \wedge, \vee, \Rightarrow, \forall, \exists$

Axiom (scheme) $A \to A$

Derivation rules - elimination and structural rules.

Each rule formalizes some proof step

Notations

Upper case latin letters denote formuli

x – bound variable

a – free variable

t – term

Γ, Φ, Λ, Π – conjunctive/disjunctive sequences of formuli

Inference rules I

Axiom

$$\frac{}{A \vdash A} \quad (I)$$

$$\frac{\Gamma \vdash \Delta, A \qquad A, \Sigma \vdash \Pi}{\Gamma, \Sigma \vdash \Delta, \Pi} \quad (Cut)$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \quad (\wedge L_1)$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta} \quad (\vee R_1)$$

$$\frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \quad (\wedge L_2)$$

$$\frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta} \quad (\vee R_2)$$

Inference rules II

$$\frac{\Gamma, A \vdash \Delta \qquad \Sigma, B \vdash \Pi}{\Gamma, \Sigma, A \vee B \vdash \Delta, \Pi} \ (\vee L)$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Sigma \vdash B, \Pi}{\Gamma, \Sigma \vdash A \wedge B, \Delta, \Pi} \ (\wedge R)$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Sigma, B \vdash \Pi}{\Gamma, \Sigma, A \rightarrow B \vdash \Delta, \Pi} \ (\rightarrow L)$$

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \rightarrow B, \Delta} \ (\rightarrow R)$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \ (\neg L)$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta} \ (\neg R)$$

Inference rules III

$$\frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x A \vdash \Delta} \quad (\forall L)$$

$$\frac{\Gamma \vdash A[y/x], \Delta}{\Gamma \vdash \forall x A, \Delta} \quad (\forall R)$$

$$\frac{\Gamma, A[y/x] \vdash \Delta}{\Gamma, \exists x A \vdash \Delta} \quad (\exists L)$$

$$\frac{\Gamma \vdash A[t/x], \Delta}{\Gamma \vdash \exists x A, \Delta} \quad (\exists R)$$

In rules $\forall R$ and $\exists L$ the variable y must not occur free within Γ and Δ. Alternatively, the variable y must not appear anywhere in the respective lower sequents.

Structural rules

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \; (WL)$$

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \; (WR)$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \; (CL)$$

$$\frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta} \; (CR)$$

$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, B, A, \Gamma_2 \vdash \Delta} \; (PL)$$

$$\frac{\Gamma \vdash \Delta_1, A, B, \Delta_2}{\Gamma \vdash \Delta_1, B, A, \Delta_2} \; (PR)$$

Inference Example

$$\dfrac{}{B \vdash B}\,(I) \qquad \dfrac{}{C \vdash C}\,(I)$$

$$\dfrac{B \vdash B \qquad C \vdash C}{B \vee C \vdash B, C}\,(\vee L)$$

$$\dfrac{B \vee C \vdash B, C}{B \vee C \vdash C, B}\,(PR)$$

$$\dfrac{B \vee C \vdash C, B}{B \vee C, \neg C \vdash B}\,(\neg L) \qquad \dfrac{}{\neg A \vdash \neg A}\,(I)$$

$$\dfrac{B \vee C, \neg C \vdash B \qquad \neg A \vdash \neg A}{(B \vee C), \neg C, (B \to \neg A) \vdash \neg A}\,(\to L)$$

$$\dfrac{(B \vee C), \neg C, (B \to \neg A) \vdash \neg A}{(B \vee C), \neg C, ((B \to \neg A) \wedge \neg C) \vdash \neg A}\,(\wedge L_1)$$

$$\dfrac{(B \vee C), \neg C, ((B \to \neg A) \wedge \neg C) \vdash \neg A}{(B \vee C), ((B \to \neg A) \wedge \neg C), \neg C \vdash \neg A}\,(PL)$$

$$\dfrac{(B \vee C), ((B \to \neg A) \wedge \neg C), \neg C \vdash \neg A}{(B \vee C), ((B \to \neg A) \wedge \neg C), ((B \to \neg A) \wedge \neg C) \vdash \neg A}\,(\wedge L_2)$$

$$\dfrac{(B \vee C), ((B \to \neg A) \wedge \neg C), ((B \to \neg A) \wedge \neg C) \vdash \neg A}{(B \vee C), ((B \to \neg A) \wedge \neg C) \vdash \neg A}\,(CL)$$

$$\dfrac{(B \vee C), ((B \to \neg A) \wedge \neg C) \vdash \neg A}{((B \to \neg A) \wedge \neg C), (B \vee C) \vdash \neg A}\,(PL)$$

$$\dfrac{}{A \vdash A}\,(I) \qquad \dfrac{A \vdash A}{\vdash \neg A, A}\,(\neg R) \qquad \dfrac{\vdash \neg A, A}{\vdash A, \neg A}\,(PR)$$

$$\dfrac{\vdash A, \neg A \qquad ((B \to \neg A) \wedge \neg C), (B \vee C) \vdash \neg A}{((B \to \neg A) \wedge \neg C), (A \to (B \vee C)) \vdash \neg A, \neg A}\,(\to L)$$

$$\dfrac{((B \to \neg A) \wedge \neg C), (A \to (B \vee C)) \vdash \neg A, \neg A}{((B \to \neg A) \wedge \neg C), (A \to (B \vee C)) \vdash \neg A}\,(CR)$$

$$\dfrac{((B \to \neg A) \wedge \neg C), (A \to (B \vee C)) \vdash \neg A}{(A \to (B \vee C)), ((B \to \neg A) \wedge \neg C) \vdash \neg A}\,(PL)$$

$$\dfrac{(A \to (B \vee C)), ((B \to \neg A) \wedge \neg C) \vdash \neg A}{(A \to (B \vee C)) \vdash (((B \to \neg A) \wedge \neg C) \to \neg A)}\,(\to R)$$

$$\dfrac{(A \to (B \vee C)) \vdash (((B \to \neg A) \wedge \neg C) \to \neg A)}{\vdash ((A \to (B \vee C)) \to (((B \to \neg A) \wedge \neg C) \to \neg A))}\,(\to R)$$