1. Show that $\mathbb{Z}$ fails to be a group under multiplication.

   Solution. $0 \in \mathbb{Z}$ is not invertible, hence $(\mathbb{Z}, \times)$ is not a group.

2. Show that $\mathbb{Z} \setminus \{0\}$ fails to be a group under multiplication.

   Solution. $2 \in \mathbb{Z} \setminus \{0\}$ is not invertible, hence $(\mathbb{Z} \setminus \{0\}, \times)$ is not a group.

3. Show that $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is a group under addition operation defined by $(a, b) + (c, d) = (a + c, b + d)$.

   Solution. The binary operation defined above is clearly associative, since the addition in $\mathbb{R}$ is associative. Element $(0, 0) \in \mathbb{R}^2$ is the identity element under addition, since

$$\forall (a, b) \in \mathbb{R}^2 : (a, b) + (0, 0) = (a + 0, b + 0) = (a, b) \ \ .$$

   Every element $(a, b) \in \mathbb{R}^2$ has an inverse $(-a, -b) \in \mathbb{R}^2$. Observe that

$$(a, b) + (-a, -b) = (a - a, b - b) = (0, 0) \ \ .$$

   It can be seen that $\mathbb{R}^2$ is closed under addition. Observe that

$$\forall (a, b), (c, d) \in \mathbb{R}^2 : (a, b) + (c, d) = (\underbrace{a + c}_{\in \mathbb{R}}, \underbrace{b + d}_{\in \mathbb{R}}) \in \mathbb{R}^2 \ \ .$$

4. What is the order of group $U(12)$ (the group of units)?

   Solution. The group of units is a multiplicative group. Since $U(12)$ is a group, every element $a \in U(12)$ must be invertible. An element a has a multiplicative inverse modulo n iff $\gcd(a, n) = 1$. The function that tells us how many numbers are co-prime to a given n is the Euler's totient function $\phi(n)$. Hence,

$$\operatorname{ord} U(12) = \phi(12) = \phi(4 \cdot 3) = \phi(4) \cdot \phi(3) = 4 \cdot \left(1 - \frac{1}{2}\right) \cdot (3 - 1) = 2 \cdot 2 = 4 \ \ .$$

   It can be see that there are only 4 elements, namely $1, 5, 7, 11$ that are co-prime to 12.

5. Is $\{0, 2\}$ a subgroup of $\mathbb{Z}_4$?

   Solution. Yes, the set $H = \{0, 2\}$ is a subgroup of $\mathbb{Z}_4$ under addition. The set H contains an identity element $0 \in H$. Element 0, as any identity, is the inverse of itself, and element 2 is also an inverse of itself, since $2 + 2 = 4 \equiv 0 \pmod 4$. It can be seen that H is closed under addition, since

$$\begin{aligned} 0 + 0 &= 0 \in H \ \ , & 0 + 2 &= 2 \in H \ \ , \\ 2 + 0 &= 2 \in H \ \ , & 2 + 2 &= 0 \in H \ \ . \end{aligned}$$

   Hence, $H = \{0, 2\}$ is a subgroup of $\mathbb{Z}_4$ under addition.

6. What are the subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$?

Solution. Let's inspect the cyclic subgroups generated by elements of

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0),(0,1),(1,0),(1,1)\} \ .$$

First, let's list the cyclic subgroups generated by elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

$$\begin{aligned} H_1 &= \langle (0,0) \rangle = \{(0,0)\} \ , \\ H_2 &= \langle (0,1) \rangle = \{(0,1),(0,0)\} \ , \\ H_2 &= \langle (1,0) \rangle = \{(1,0),(0,0)\} \ , \\ H_4 &= \langle (1,1) \rangle = \{(1,1),(0,0)\} \ . \end{aligned}$$

Hence, group $\mathbb{Z}_2 \times \mathbb{Z}_2$ contains at least subgroups H_1, H_2, H_3, H_4.

Next, we inspect 3-element subsets of $\mathbb{Z}_2 \times \mathbb{Z}_2$. We are interested in the ones that contain the identity $(0,0)$ element (otherwise it won't be a subgroup). There are 3 such subsets:

$$\begin{aligned} H_5 &= \{(0,0),(0,1),(1,0)\} \ , \\ H_6 &= \{(0,0),(0,1),(1,1)\} \ , \\ H_7 &= \{(0,0),(1,0),(1,1)\} \ . \end{aligned}$$

It can be seen that H_5 is not a subgroup, since it is not closed under addition:

$$(0,1) + (1,0) = (1,1) \notin H_5 \ .$$

For the same reason, H_6 is not a subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Observe that

$$(0,1) + (1,1) = (1,0) \notin \mathbb{Z}_6 \ .$$

The set H_7 is also not closed under addition, since

$$(1,0) + (1,1) = (0,1) \notin H_7 \ .$$

Hence, the subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$ are only the 4 cyclic subgroups H_1, H_2, H_3, H_4 generated by elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

7. Show that $\{-1, 1, i, -i\}$ is a subgroup of $(\mathbb{C} \setminus \{0\}, \times)$.

Solution. The fact that $\{-1, 1, i, -i\}$ is a subgroup of $(\mathbb{C} \setminus \{0\}, \times)$ can be seen by inspecting Table 1 – the Cayley table for this subgroup.

Table 1: Cayley table for the subgroup $\{-1, 1, i, -i\}$ of $(\mathbb{C} \setminus \{0\}, \times)$.

$\times$	-1	1	i	$-i$
-1	1	-1	$-i$	i
1	-1	1	i	$-i$
i	$-i$	i	-1	1
$-i$	i	$-i$	1	-1

8. Is $\mathbb{Z}$ a cyclic group?

   Solution. Yes, it is. It can be seen that

$$
\begin{aligned}
\mathbb{Z} &= \{\ldots, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, \ldots\} \ , \\
\langle 1 \rangle &= \{\ldots, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, \ldots\} \ , \\
\langle -1 \rangle &= \{\ldots, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, \ldots\} \ .
\end{aligned}
$$

   Hence, $\mathbb{Z}$ is generated by 1 and -1, and therefore is cyclic.

9. Show that $\mathbb{Z}_6$ is generated by both 1 and 5.

   Solution.

$$
\begin{aligned}
\mathbb{Z}_6 &= \{0, 1, 2, 3, 4, 5\} \ , \\
\langle 1 \rangle &= \{1, 2, 3, 4, 5, 0\} \ , \\
\langle 5 \rangle &= \{5, 4, 3, 2, 1, 0\} \ .
\end{aligned}
$$

   Hence, $\mathbb{Z}_6$ is generated by 1 and 5.

10. Is $3\mathbb{Z}$ a cyclic subgroup of $\mathbb{Z}$?

    Solution. Yes, it is. It can be seen that

$$
\begin{aligned}
\mathbb{Z} &= \{\ldots, -12, -9, -6, -3, 0, 3, 6, 9, 12, \ldots\} \ , \\
\langle 3 \rangle &= \{\ldots, -12, -9, -6, -3, 0, 3, 6, 9, 12, \ldots\} \ , \\
\langle -3 \rangle &= \{\ldots, -12, -9, -6, -3, 0, 3, 6, 9, 12, \ldots\} \ .
\end{aligned}
$$

    Hence, $3\mathbb{Z}$ is generated by 3 and -3, and since $3\mathbb{Z} \subset \mathbb{Z}$ it is a subgroup of $\mathbb{Z}$.

11. What is the order of 4 in $\mathbb{Z}_6$?

    Solution. It can be seen that element $4 \in \mathbb{Z}_6$ generates a cyclic subgroup $\langle 4 \rangle = \{4, 2, 0\}$ of order 3, and hence the order of 4 is 3. It can be seen that $3 \cdot 4 = 12 \equiv 0 \pmod 6$, and hence 3 is the minimal integer m, such that 4^m is an identity in $\mathbb{Z}_6$.

12. What is the order of 2 in $\mathbb{Z}_5$? Does 2 generate $\mathbb{Z}_5$?

    Solution. Yes, group $\mathbb{Z}_5$ is generated by 2. Since 2 is a generator, its order is 5. Observe that $\langle 2 \rangle = \{2, 4, 1, 3, 0\} = \mathbb{Z}_5$.

13. What is the order of 2 in $U(5)$?

    Solution. Element $2 \in U(5)$ generates a cyclic subgroup $\langle 2 \rangle = \{2, 4, 3, 1\}$ of order 4, hence the order of 2 is 4 in $U(5)$. It can also be seen that $2^4 = 16 \equiv 1 \pmod 5$.

14. What is the order of 5 in $U(12)$?

Solution. Element 5 generates a cyclic subgroup $\langle 5\rangle = \{5, 1\}$ of order 2, and so the order of 5 is 2 in $U(12)$.

15. What is the order of $-i \in \mathbb{C} \setminus \{0\}$?

    Solution. Element $-i$ generates a cyclic subgroup $\langle -i\rangle = \{-i, -1, i, 1\}$ of order 4, hence the order of $-i$ is 4 in $\mathbb{C} \setminus \{0\}$.

16. What is the group structure of $U(9)$? Is $U(9)$ a cyclic group?

    Solution. Group $U(9) = \{1, 2, 4, 5, 7, 8\}$ contains the following cyclic subgroups:

$$\begin{aligned} &\langle 1\rangle = \{1\} \ , && \langle 2\rangle = \{2, 4, 8, 7, 5, 1\} = U(9) \ , \\ &\langle 4\rangle = \{4, 7, 1\} \ , && \langle 5\rangle = \{5, 7, 8, 4, 2, 1\} = U(9) \ , \\ &\langle 7\rangle = \{7, 4, 1\} \ , && \langle 8\rangle = \{8, 1\} \ . \end{aligned}$$

    Group $U(9)$ is generated by 2 and 5, and hence is cyclic. The structure is the following:

    - 1 element of order 1 (identity)
    - 1 element of order 2
    - 2 elements of order 3
    - 2 elements of order 6 (generators)

17. What is the group structure of $U(8)$? Is $U(8)$ a cyclic group?

    Solution. Group $U(8) = \{1, 3, 5, 7\}$ contains the following cyclic subgroups:

$$\begin{aligned} &\langle 1\rangle = \{1\} \ , && \langle 3\rangle = \{3, 1\} \ , \\ &\langle 5\rangle = \{5, 1\} \ , && \langle 7\rangle = \{7, 1\} \ . \end{aligned}$$

    Since $U(8)$ does not have any generators, it is not cyclic. The group structure is the following:

    - 1 element of order 1 (identity)
    - 3 elements of order 2

18. If $a^{24} = e$ in group G, what are possible orders of a?

    Solution. By definition, the order of a is the minimal integer n such that $a^n = e \in G$. If $a^{24} = e$, then 24 is a multiple of the order of a. Hence, the order of a is any divisor of $24 : 1, 2, 3, 4, 6, 8, 12, 24$.

19. Suppose G is a finite group with an element g with order 5, and and an element h of order 7. What are possible orders of G?

    Solution. By the Lagrange's theorem, the order of any element must divide the order of the group. Let the order of group G be a. Then 5 and 7 must be the divisors of a. Hence, the order of the group is greater or equal to the least common multiple of 5 and 7. Hence, ord $G \geqslant 35$.

20. Show that $U(8)$ and $\mathbb{Z}_4$ have different group structures.

Solution. Group $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ contains the following cyclic subgroups:

$$\langle 0 \rangle = \{0\} \ , \qquad \langle 1 \rangle = \{1, 2, 3, 0\} = \mathbb{Z}_4 \ ,$$
$$\langle 2 \rangle = \{2, 0\} \ , \qquad \langle 3 \rangle = \{3, 2, 1, 0\} = \mathbb{Z}_4 \ .$$

Group $U(8) = \{1, 3, 5, 7\}$ consists of the following cyclic subgroups:

$$\langle 1 \rangle = \{1\} \ , \qquad \langle 3 \rangle = \{3, 1\} \ ,$$
$$\langle 5 \rangle = \{5, 1\} \ , \qquad \langle 7 \rangle = \{7, 1\} \ .$$

Clearly, these two groups have different structure. Group $\mathbb{Z}_4$ is cyclic, while group $U(8)$ is not. For comparison, observe that $U(8)$ has the same structure as $\mathbb{Z}_2 \times \mathbb{Z}_2$.

21. Show that $U(5)$ and $U(10)$ have the same group structure, but not $U(12)$.

Solution. Group $U(5) = \{1, 2, 3, 4\}$ consists of the following cyclic subgroups

$$\langle 1 \rangle = \{1\} \ , \qquad \langle 2 \rangle = \{2, 4, 3, 1\} = U(5) \ ,$$
$$\langle 3 \rangle = \{3, 4, 2, 1\} = U(5) \ , \qquad \langle 4 \rangle = \{4, 1\} \ .$$

This shows that group $U(5)$ is generated by 5 (and therefore is cyclic), has 1 element of order 1, 1 element of order 2, 1 element of order 3 and 1 element of order 4.

Group $U(10) = \{1, 3, 7, 9\}$ consists of the following cyclic subgroups

$$\langle 1 \rangle = \{1\} \ , \qquad \langle 3 \rangle = \{3, 9, 7, 1\} = U(10) \ ,$$
$$\langle 7 \rangle = \{7, 9, 3, 1\} = U(10) \ , \qquad \langle 9 \rangle = \{9, 1\} \ .$$

Group $U(12) = \{1, 5, 7, 11\}$ consists of the following cyclic subgroups

$$\langle 1 \rangle = \{1\} \ , \qquad \langle 5 \rangle = \{5, 1\} \ ,$$
$$\langle 7 \rangle = \{7, 1\} \ , \qquad \langle 11 \rangle = \{11, 1\} \ .$$

It can be seen that indeed, groups $U(5)$ and $U(10)$ share the same group structure – both are cyclic groups containing 1 element of order 1 (identity), 2 elements of order 4 (generators), and 1 element of order 2.

It can be seen that the group structure of $U(12)$ is different from that of $U(5)$ and $U(10)$. $U(12)$ is not a cyclic group. Observe that $U(12)$ has the same structure as $U(8)$, and the same structure as $\mathbb{Z}_2 \times \mathbb{Z}_2$ (task 6).

22. Do groups $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ have the same group structure?

Solution. Group $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ consists of the following cyclic subgroups

$$\langle 0 \rangle = \{0\} \ , \qquad \langle 1 \rangle = \{1, 2, 3, 0\} = \mathbb{Z}_4 \ ,$$
$$\langle 2 \rangle = \{2, 0\} \ , \qquad \langle 3 \rangle = \{3, 2, 1, 0\} = \mathbb{Z}_4 \ .$$

The structure of $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ is known to us from the previous tasks (task 6, specifically). This group consists of 1 element of order 1 and 3 elements of order 2.

Clearly, $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ have different structure. Group $\mathbb{Z}_4$ is cyclic, while $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not.

23. Do groups $\mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ have the same group structure?

Solution. Group $\mathbb{Z}_8 = \{0,1,2,3,4,5,6,7\}$ consists of the following cyclic subgroups

$$\begin{aligned}
&\langle 0 \rangle = \{0\} \ , && \langle 1 \rangle = \{1,2,3,4,5,6,7,0\} = \mathbb{Z}_8 \ , \\
&\langle 2 \rangle = \{2,4,6,0\} \ , && \langle 3 \rangle = \{3,6,1,4,7,2,5,0\} = \mathbb{Z}_8 \ , \\
&\langle 4 \rangle = \{4,0\} \ , && \langle 5 \rangle = \{5,2,7,4,1,6,3,0\} = \mathbb{Z}_8 \ , \\
&\langle 6 \rangle = \{6,4,2,0\} \ , && \langle 7 \rangle = \{7,6,5,4,3,2,1,0\} = \mathbb{Z}_8 \ .
\end{aligned}$$

Group $\mathbb{Z}_4 \times \mathbb{Z}_2 = \{(0,0),(0,1),(1,0),(1,1),(2,0),(2,1),(3,0),(3,1)\}$ consists of the following cyclic subgroups

$$\begin{aligned}
&\langle (0,0) \rangle = \{(0,0)\} \ , && \langle (0,1) \rangle = \{(0,1),(0,0)\} \ , \\
&\langle (1,0) \rangle = \{(1,0),(2,0),(3,0),(0,0)\} \ , && \langle (1,1) \rangle = \{(1,1),(2,0),(3,1),(0,0)\} \ , \\
&\langle (2,0) \rangle = \{(2,0),(0,0)\} \ , && \langle (2,1) \rangle = \{(2,1),(0,0)\} \ , \\
&\langle (3,0) \rangle = \{(3,0),(2,0),(1,0),(0,0)\} \ , && \langle (3,1) \rangle = \{(3,1),(2,0),(1,1),(0,0)\} \ .
\end{aligned}$$

Group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}$ consists of the following cyclic subgroups

$$\begin{aligned}
&\langle (0,0,0) \rangle = \{(0,0,0)\} \ , && \langle (0,0,1) \rangle = \{(0,0,1),(0,0,0)\} \ , \\
&\langle (0,1,0) \rangle = \{(0,1,0),(0,0,0)\} \ , && \langle (0,1,1) \rangle = \{(0,1,1),(0,0,0)\} \ , \\
&\langle (1,0,0) \rangle = \{(1,0,0),(0,0,0)\} \ , && \langle (1,0,1) \rangle = \{(1,0,1),(0,0,0)\} \ , \\
&\langle (1,1,0) \rangle = \{(1,1,0),(0,0,0)\} \ , && \langle (1,1,1) \rangle = \{(1,1,1),(0,0,0)\} \ .
\end{aligned}$$

It can be easily seen that group $\mathbb{Z}_8$ is cyclic and has as much as 4 generators, while groups $\mathbb{Z}_4 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ are not cyclic. Group $\mathbb{Z}_4 \times \mathbb{Z}_2$ has an element of order 4, while group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ has no element of order 4. Hence, all three groups are different.