Exercise 1. Determine which of the following functions are injective and which are surjective.

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x$.

(b) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = n^2 + 3$.

(c) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin x$.

(d) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x^2$.

(e) $f : \mathbb{Z} \to \mathbb{Q}$ defined by $f(n) = n/1$.

(f) $f : \mathbb{Q} \to \mathbb{Z}$ defined by $f(p/q) = p$, where p/q is a rational number expressed in its lowest terms with a positive denominator.

(g) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 2x$.

Solution.

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x$ is injective, since $\forall a, b \in \mathbb{R} : e^a = e^b \implies a = b$. f is not surjective, since there is no $x \in \mathbb{R}$ such that $f(x) = -1 \in \mathbb{R}$.

(b) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = n^2 + 3$ is not injective, since $f(n) = f(-n) = n^2 + 3$. f is not surjective, since there is no $n \in \mathbb{Z}$ such that $f(n) = 5 \in \mathbb{Z}$.

$$n^2 + 3 = 5 \implies n^2 = 2 \implies n = \pm\sqrt{2} \notin \mathbb{Z} \ .$$

(c) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin x$ is not injective, since $f(\pi) = f(2\pi) = 0$. f is not surjective, since there is no $x \in \mathbb{R}$ such that $f(x) = 2 \in \mathbb{R}$.

(d) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x^2$ is not injective, since $f(1) = f(-1) = 1$. f is not surjective, since there is no $x \in \mathbb{Z}$ such that $f(x) = -1 \in \mathbb{Z}$

(e) $f : \mathbb{Z} \to \mathbb{Q}$ be defined by $f(n) = n/1$ is injective, since $\forall a, b \in \mathbb{Z} : a/1 = b/1 \implies a = b$. f is not surjective, since there is no $n \in \mathbb{Z}$ such that $f(n) = \frac{1}{2} \in \mathbb{Q}$.

(f) $f : \mathbb{Q} \to \mathbb{Z}$ be defined by $f(p/q) = p$ where p/q is a rational number expressed in its lowest terms with a positive denominator is not injective, since $f\left(\frac{1}{2}\right) = f\left(\frac{1}{3}\right) = 1$. f is surjective, since

$$\forall y \in \mathbb{Z} : \exists x = \frac{y}{1} \in \mathbb{Q} : f\left(\frac{y}{1}\right) = y \ .$$

(g) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 2x$ is injective, since $2x = 2y \implies x = y$. f is not surjective, since there is no $x \in \mathbb{Z}$ such that $f(x) = 3 \in \mathbb{Z}$.

Exercise 2. Is relation $f \subseteq \mathbb{Q} \times \mathbb{Z}$ given by $f\left(\frac{p}{q}\right) = p$ a mapping?

Solution. The relation $f : \mathbb{Q} \times \mathbb{Z}$ given by $f\left(\frac{p}{q}\right) = p$ is not a mapping, since $\frac{1}{2} = \frac{2}{4} \in \mathbb{Q}$, but $1 = f\left(\frac{1}{2}\right) \neq f\left(\frac{2}{4}\right) = 2$, thus making f not functional, and hence violating the definition of a mapping.

Exercise 3. Is the relation $f \subseteq \mathbb{Z} \times \mathbb{Z}$ defined by $f(x) = 2x$ a mapping?

Solution. f is left-total and functional, since $\forall x \in \mathbb{Z} : \exists 2x \in \mathbb{Z}$. Therefore, f is a mapping.

Exercise 4. Is the relation $f \subseteq \mathbb{Z} \times \mathbb{Z}$ defined by $f(x) = \frac{x}{2}$ a mapping?

Solution. f is not left-total, since f does not map $3 \in \mathbb{Z}$, as $f(3) \in \mathbb{Z}$. f is functional, since $a \in \mathbb{Z}$ is uniquely mapped to $\frac{a}{2} \in \mathbb{Z}$. Since f is not left-total (f is defined for even integers only), it is not a mapping.

Exercise 5. Which of the following relations $f \subseteq \mathbb{Q} \times \mathbb{Q}$ define a mapping? If f is not a mapping, supply a reason for it.

$$(a) \quad f\left(\frac{p}{q}\right) = \frac{p+1}{p-2} \qquad\qquad (b) \quad f\left(\frac{p}{q}\right) = \frac{3p}{2q}$$
$$(c) \quad f\left(\frac{p}{q}\right) = \frac{p+q}{q^2} \qquad\qquad (d) \quad f\left(\frac{p}{q}\right) = \frac{3p^2}{7q^2} - \frac{p}{q}$$

Solution.

(a) f is not a mapping. f is not left-total, since $f(2/3)$ is undefined. f is not right-unique, since

$$-\frac{2}{3} = f\left(\frac{1}{3}\right) \neq f\left(\frac{3}{9}\right) = 4 \ .$$

(b) f is a mapping. f is left-total, since

$$\forall \frac{p}{q} \in \mathbb{Q} : \exists \frac{3p}{2q} \in \mathbb{Q} \ .$$

f is functional since for every $\frac{p}{q} \in \mathbb{Q}$ there exist unique element $\frac{3p}{2q} \in \mathbb{Q}$. Consider an equivalent element $\frac{p'}{q'} \in \mathbb{Q}$ such that $p' = kp, q' = kq$, and $k \in \mathbb{Z}$. Then

$$f\left(\frac{p'}{q'}\right) = \frac{3 \cdot k \cdot p}{2 \cdot k \cdot q} = \frac{3p}{2q} = f\left(\frac{p}{q}\right) \ .$$

Hence, $f\left(\frac{p}{q}\right)$ as well as $f\left(\frac{p'}{q'}\right)$ correspond to the same element $\frac{3p}{2q}$. Therefore, f is functional.

(c) f is not a mapping. f is left-total

$$\forall \frac{p}{q} \in \mathbb{Q} : \exists \frac{p+q}{q^2} \in \mathbb{Q} \ .$$

However, f is not functional, since $\frac{1}{2} = \frac{2}{4}$, but

$$\frac{3}{4} = f\left(\frac{1}{2}\right) \neq f\left(\frac{2}{4}\right) = \frac{3}{8} \ .$$

(c) f is a mapping. f is left-total

$$\forall \frac{p}{q} \in \mathbb{Q}: \exists \frac{3p^2}{7q^2} - \frac{p}{q} \in \mathbb{Q} \ .$$

f is functional, since for every element $\frac{p}{q} \in \mathbb{Q}$ there exists a unique element $\frac{3p^2}{7q^2} - \frac{p}{q}$. Consider an equivalent element $\frac{p'}{q'}$ such that $p' = kp, q' = kq$, and $k \in \mathbb{Z}$. Then

$$f\left(\frac{p'}{q'}\right) = \frac{3(kp)^2}{7(kq)^2} - \frac{kp}{kq} = \frac{3k^2p^2 - 7k^2qp}{7k^2q^2} = \frac{3p^2 - 7qp}{7q^2} = \frac{3p^2}{7q^2} - \frac{p}{q} = f\left(\frac{p}{q}\right) \ .$$

Hence, $f\left(\frac{p}{q}\right)$ as well as $f\left(\frac{p'}{q'}\right)$ correspond do the same element $\frac{3p^2}{7q^2} - \frac{p}{q}$. Therefore, f is functional.