Exercise 1. Given the two sets

$$A = \{x \in \mathbb{R} : 0 < x \leqslant 3\} \ ,$$
$$B = \{x \in \mathbb{R} : 2 \leqslant x < 4\} \ ,$$

define the following sets: $A \cap B$; $A \cup B$; $A \setminus B$; A'.

Solution.

$$A \cap B = \{x \in \mathbb{R} : 2 \leqslant x \leqslant 3\}$$
$$A \cup B = \{x \in \mathbb{R} : 0 < x < 4\}$$
$$A \setminus B = \{x \in \mathbb{R} : 0 < x < 2\}$$
$$A' = \{x \in \mathbb{R} : x \leqslant 0 \vee x > 3\}$$

Exercise 2. Suppose that

$$A = \{x \in \mathbb{N} : x \text{ is even}\} \ ,$$
$$B = \{x \in \mathbb{N} : x \text{ is prime}\} \ ,$$
$$C = \{x \in \mathbb{N} : x \text{ is a multiple of } 5\} \ .$$

Describe each of the following sets.

(a) $A \cap B$ (b) $B \cap C$

(c) $A \cup B$ (d) $A \cap (B \cup C)$

Solution.

1. (a) $A \cap B = \{x \in \mathbb{N} : x \text{ is even} \wedge x \text{ is prime}\} = \{2\}$.
2. (b) $B \cap C = \{x \in \mathbb{N} : x \text{ is prime} \wedge x \text{ is a multiple of } 5\} = \{5\}$.
3. (c) $A \cup B = \{x \in \mathbb{N} : x \text{ is even} \vee x \text{ is prime}\}$.
4. (d) $A \cap (B \cup C) = \{x \in \mathbb{N} : x \text{ is even} \wedge (x \text{ is prime} \vee x \text{ is a multiple of } 5)\}$.

Exercise 3. If $A = \{a, b, c\}, B = \{1, 2, 3\}, C = \{x\}$, and $D = \emptyset$, list all of the elements in each of the following sets.

(a) $A \times B$ (b) $B \times A$

(c) $A \times B \times C$ (d) $A \times D$

Solution.

(a) $A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3), (c, 1), (c, 2), (c, 3)\}$.

(b) $B \times A = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c)\}$.

(c) $A \times B \times C = \{(a, 1, x), (a, 2, x), (a, 3, x), (b, 1, x), (b, 2, x), (b, 3, x), (c, 1, x), (c, 2, x), (c, 3, x)\}$.

(d) $A \times D = \{(a,b) : \underbrace{a \in A \wedge b \in \emptyset}_{\text{FALSE}}\} = \emptyset$.

Exercise 4. Let A be a set. Show that $A \cap A = A$.

Solution.

$$A \cap A = \{x : x \in A \wedge x \in A\} = \{x \in A\} = A \ .$$

Exercise 5. Let A be a set. Show that $A \cup \emptyset = A$.

Solution.

$$A \cup \emptyset = \{x : x \in A \vee x \in \emptyset\} = \{x : x \in A\} = A \ .$$

Exercise 6. Let A, B, C be sets. Show that $A \cup (B \cup C) = (A \cup B) \cup C$.

Solution.

$$\begin{aligned} A \cup (B \cup C) &= A \cup \{x : x \in B \vee x \in C\} \\ &= \{x : x \in A \vee x \in B \vee x \in C\} \\ &= \{x : x \in A \vee x \in B\} \cup C \\ &= (A \cup B) \cup C \ . \end{aligned}$$

Exercise 7. Let A, B be sets. Show that $A \cup B = B \cup A$.

Solution.

$$A \cup B = \{x : x \in A \vee x \in B\} = B \cup A \ .$$

Exercise 8. Let A, B, C be sets. Show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Solution.

$$\begin{aligned} A \cup (B \cap C) &= A \cup \{x : x \in B \wedge x \in C\} \\ &= \{x \in A \vee (x \in B \wedge x \in C)\} \\ &= \{(x \in A \vee x \in B) \wedge (x \in A \vee x \in C)\} \\ &= (A \cup B) \cap (A \cup C) \ . \end{aligned}$$

Exercise 9. Let A, B be sets. Show that $(A \cup B)' = A' \cap B'$.

Solution. We must show that $(A \cup B)' \subseteq A' \cap B'$ and $(A \cup B)' \supseteq A' \cap B'$.

$$x \in (A \cup B)' \implies x \notin (A \cup B) \implies x \notin A \wedge x \notin B \implies x \in A' \wedge x \in B' \implies x \in A' \cap B' \ .$$
$$x \in A' \cap B' \implies x \in A' \wedge x \in B' \implies x \notin A \wedge x \notin B \implies x \notin A \cup B \implies x \in (A \cup B)' \ .$$

Therefore, $(A \cup B)' \subseteq A' \cap B'$ and $(A \cup B)' \supseteq A' \cap B'$. Hence, $(A \cup B)' = A' \cap B'$.

Exercise 10. Let A, B, C be sets. Show that $A \cup B = (A \cap B) \cup (A \setminus B) \cup (B \setminus A)$.

Solution.

$$\begin{aligned} A \cup B &= (A \cap B) \cup (A \setminus B) \cup (B \setminus A) \\ &= (A \cap B) \cup (A \cap B') \cup (B \cap A') \\ &= (A \cap (B \cup B')) \cup (B \cap A') \\ &= A \cup (B \cap A') = (A \cup B) \cap (A \cup A') \\ &= (A \cup B) = A \cup B \ . \end{aligned}$$