

## Definitions

**Definition 1** (Subset). Set  $A$  is a subset of a set  $B$  (written  $A \subseteq B$ ) if  $a \in A \implies a \in B$ .

**Definition 2** (Equality of sets). Sets  $A$  and  $B$  are equal (written  $A = B$ ) if  $A \subseteq B \wedge B \subseteq A$ .

**Definition 3** (Proper subset). Set  $A$  is a proper subset of  $B$  (written  $A \subset B$ ) if  $A \subseteq B \wedge A \neq B$ .

**Definition 4** (Empty set). Emptysset ( $\emptyset$ ) is defined as a set containing no elements:  $\forall x : x \notin \emptyset$ .

**Definition 5** (Union of sets). Union of sets  $A$  and  $B$  is a set  $A \cup B = \{x : x \in A \vee x \in B\}$ .

**Definition 6** (Intersection of sets). Intersection of sets  $A$  and  $B$  is a set  $A \cap B = \{x : x \in A \wedge x \in B\}$ .

**Definition 7** (Disjoint sets). Sets  $A$  and  $B$  are disjoint if  $A \cap B = \emptyset$ .

**Definition 8** (Set compliment). Let  $U$  be the universal class, and let  $A \subset U$ . The compliment of  $A$  is the set  $A' = \{x \in U : x \notin A\}$ .

**Definition 9** (Set difference). The difference of sets  $A$  and  $B$  is the set  $A \setminus B = A \cap B' = \{x \in A : x \notin B\}$ .

**Definition 10** (Cartesian product of sets). Cartesian product of sets  $A$  and  $B$  is the set  $A \times B = \{(a, b) : a \in A \wedge b \in B\}$ .

**Definition 11** (Binary Relation). Relation  $R$  is a binary relation between sets  $A$  and  $B$  if  $R \subseteq A \times B$ .

The notation  $xRy$  means  $(x, y) \in R$ . Let  $R \subseteq A \times B$ . The set  $A$  is called the domain of  $R$ , and the set  $B$  is the co-domain of  $R$ .

**Definition 12** (Endorelation). Relation  $R$  is an endorelation on set  $A$  if  $R \subseteq A^2 = A \times A$ .

**Definition 13** (Image of a set under a binary relation). Let  $R \subseteq A \times B$  be a binary relation. Then the image of  $A$  under  $R$  is the set  $Im(R) = \{y \in B : \exists x \in A : xRy\}$ .

**Definition 14** (Preimage of a set under binary relation). Let  $R \subseteq A \times B$  be a binary relation, and let  $Y \subseteq B$ . Then the preimage of  $Y$  under  $R$  (written  $R^{-1}(Y)$ ) is  $R^{-1}(Y) = \{x \in A : \exists y \in Y : xRy\}$ .

**Definition 15** (Field of a binary relation). Let  $R$  be a binary relation. Then  $Field(R) = Dom(R) \cup Im(R)$ .

**Definition 16** (Injection). A binary relation  $R \subseteq A \times B$  is injective if  $\forall x, z \in A, \forall y \in B : xRy \wedge zRy \implies x = z$ .

In example, the relation  $R = \{(x, x^2)\} \subseteq \mathbb{Z} \times \mathbb{Z}$  is not injective, since both  $(2, 4)$  and  $(-2, 4)$  are in  $R$ , and hence injectivity does not hold.

**Definition 17** (Surjection). A binary relation  $R \subseteq A \times B$  is surjective if  $\forall y \in B \exists x \in A : xRy$ .

In example, the relation  $R = \{(x, x^2)\} \subseteq \mathbb{Z} \times \mathbb{Z}$  is not surjective, since no preimage exists for  $3 \in \mathbb{Z}$ , since  $\sqrt{3} \notin \mathbb{Z}$ .

**Definition 18** (Bijection). Injective and surjective binary relation is called bijective.

**Definition 19** (Set cardinality). Cardinality of a set  $A$ , denoted as  $|A|$ , is a measure of the number of elements in the set.

$|A| = |B|$  if there exists a bijection  $f : A \rightarrow B$ .

$|A| \leq |B|$  if there exists an injection  $f : A \rightarrow B$ .

$|A| < |B|$  if there exists an injection  $f : A \rightarrow B$ , but no bijection  $g : A \rightarrow B$  exists.

**Definition 20** (Countable set). Set  $A$  is countable if  $|A| = |B|$ , where  $B \subseteq \mathbb{N}$ .

**Definition 21** (Countably infinite set). Set  $A$  is countably infinite if there exists a bijection  $f : A \rightarrow \mathbb{N}$ .

**Definition 22** (Reflexivity). Binary relation  $R$  on a set  $A$  is **reflexive** if every element  $x$  in  $A$  is related to itself:  $\forall x \in A : xRx$ .

In example, relation  $\leq$  on  $\mathbb{Z}$  is reflexive, since  $\forall a \in \mathbb{Z} : a \leq a$ . However, the relation  $<$  is not reflexive, since  $a < a$  does not hold.

**Definition 23** (Anti-reflexivity). Binary relation  $R$  is called **anti-reflexive** if every element  $x$  in  $A$  is not related to itself:  $\forall x \in A : \neg(xRx)$ .

In example, relation  $<$  on  $\mathbb{Z}$  is anti-reflexive, since  $\forall a \in \mathbb{Z} : a \not< a$ .

**Definition 24** (Symmetry). Relation  $R$  on a set  $A$  is called **symmetric** if  $\forall x, y \in A : xRy \implies yRx$ .

In example, equality relation  $=$  on  $\mathbb{R}$  is symmetric, since  $\forall a, b \in \mathbb{R} : a = b \implies b = a$ .

**Definition 25** (Anti-symmetry). Relation  $R$  on a set  $A$  is **anti-symmetric** if  $\forall x, y \in A : xRy \wedge yRx \implies x = y$ .

In example, relation  $\leq$  is anti-symmetric, since  $x \leq y \wedge y \leq x \implies x = y$ .

**Definition 26** (Asymmetry). Relation  $R$  on a set  $A$  is **asymmetric** if  $\forall x, y \in A : xRy \implies \neg(yRx)$ .

In example: relation  $<$  on  $\mathbb{R}$  is asymmetric, since  $x < y \implies \neg(y < x)$ .

**Definition 27** (Transitivity). Relation  $R$  on a set  $A$  is **transitive** if  $\forall x, y, z \in A : xRy \wedge yRz \implies xRz$ .

In example, it can be seen that relations  $<$  and  $=$  are transitive

$$\begin{aligned} a < b \wedge b < c &\implies a < c , \\ a = b \wedge b = c &\implies a = c . \end{aligned}$$

**Definition 28** (Connexity). Relation  $R$  on a set  $A$  is **connex** if  $\forall x, y \in A : xRy \vee yRx$ .

**Definition 29** (Trichotomy).  $R$  is called **trichotomous** if  $\forall x, y \in A : xRy \vee yRx \vee x = y$ .

**Definition 30** (Left-total binary relation). A binary relation  $R \subseteq A \times B$  is left-total if  $\forall x \in A \exists y \in B : xRy$ .

**Definition 31** (Partial function). A binary relation  $R \subseteq A \times B$  is a partial function if  $\forall x \in A, \forall y, z \in B : xRy \wedge xRz \implies y = z$ .

In example, mapping  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  defined by  $y = \sqrt{x}$  is not a partial function, since  $2 = \sqrt{4} = -2$ .

**Definition 32** (Function, Mapping). A binary relation  $R \subseteq A \times B$  is a function (or a mapping)  $F : A \rightarrow B$  if it is left-total, injective, and is a partial function.

The notation  $a \xrightarrow{f} b$  means that element  $a \in A$  is mapped to element  $b \in B$  by mapping  $f : A \rightarrow B$ . Equivalently, the same can be expressed as  $f(a) = b$ . In other words, mapping  $F : A \rightarrow B$  maps every element  $a \in A$  to a *unique* element  $b \in B$ .

**Definition 33** (Linear Map). A **linear mapping** or **linear transformation** is a map  $\mathbb{R}^n \rightarrow \mathbb{R}^m$  given by a matrix.

In example, given a  $2 \times 2$  matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ,$$

we can define a map  $T_A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by

$$\forall (x, y) \in \mathbb{R}^2 : T_A(x, y) = (ax + by, cx + dy) .$$

This is actually matrix multiplication, that is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} .$$

**Definition 34** (Permutation). For any set  $S$ , a bijective mapping  $\pi : S \rightarrow S$  is called a **permutation**.

Suppose  $S = \{1, 2, 3\}$ . Define a map  $\pi : S \rightarrow S$  by

$$\begin{pmatrix} 1 & 2 & 3 \\ \pi(1) & \pi(2) & \pi(3) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} .$$

It is easy to verify that this map is bijective, hence this map is a permutation of  $S$ .

**Definition 35** (Identity Map). The **identity map**  $id_S$  is such that  $\forall s \in S : s \mapsto s$ .

In example, for  $S = \{1, 2, 3\}$ , the identity map  $id_S$  is

$$\begin{pmatrix} 1 & 2 & 3 \\ \pi(1) & \pi(2) & \pi(3) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} .$$

**Definition 36** (Function Composition). A **composition of function**  $f : A \rightarrow B$  and  $g : B \rightarrow C$  is a new function  $h : A \rightarrow C$  defined by

$$(g \circ f)(x) = g(f(x)) .$$

Note that  $g(f(x)) = (g \circ f)(x) \neq (f \circ g)(x) = f(g(x))$ .

For example, consider the following sets

$$A = \{1, 2, 3\} \qquad B = \{a, b, c\} \qquad C = \{x, y, z\} .$$

Consider maps

$$\begin{aligned} f : A \rightarrow B & \text{ defined by } \{1 \mapsto b, 2 \mapsto c, 3 \mapsto a\} , \\ g : B \rightarrow C & \text{ defined by } \{a \mapsto z, b \mapsto z, c \mapsto x\} . \end{aligned}$$

The composition  $g \circ f : A \rightarrow C$  is defined by  $\{1 \mapsto z, 2 \mapsto x, 3 \mapsto z\}$ .

It can be seen than the composition  $f \circ g$  is not a valid map.

**Definition 37** (Inverse Map). Let  $f : A \rightarrow B$  be a function. The **inverse map**  $f^{-1} : B \rightarrow A$  is a function such that

$$\begin{aligned} f \circ f^{-1} &= id_B , \\ f^{-1} \circ f &= id_A . \end{aligned}$$

Function  $f : \mathbb{R}^+ \rightarrow \mathbb{R}$  defined by  $f(x) = \ln(x)$  has an inverse  $f^{-1}(x) = e^x$ .

$$\begin{aligned} (f \circ f^{-1})(x) &= f(f^{-1}(x)) = f(e^x) = \ln e^x = x , \\ (f^{-1} \circ f)(x) &= f^{-1}(\ln x) = e^{\ln x} = x . \end{aligned}$$

**Definition 38** (Equivalence Relation). A binary relation  $R$  on a set  $X$  is an equivalence relation iff it is reflexive, symmetric and transitive.

**Definition 39** (Set partition). A partition of a set  $X$  is a set of non-empty subsets  $A_i \subset X$  such that they are pairwise disjoint  $A_i \cap A_j = \emptyset \forall i \neq j$ , and  $\bigcup_i A_i = X$ .

Any such subset  $A_i$  is an equivalence class under equivalence relation  $\sim$ . Every equivalence relation gives rise to a partition, and likewise if there exists a partition of some set, there exists an equivalence relation which generates this partition.

**Definition 40** (Equivalence Class). An equivalence class of  $x \in X$  under equivalence relation  $R$ , denoted as  $[x]$  is a set of elements that are equivalent under  $R : [x] = \{y \in X : x \sim y\}$ .

**Definition 41** (Factor Set). The factor set of a set  $X$  under equivalence relation  $\sim$ , denoted as  $X / \sim$ , is a set consisting of representative of each of the equivalence classes in a partition. The cardinality of the factor set is equal to the number of equivalence classes in a partition generated by  $\sim$ .

**Definition 42** (Setoid). A set with an equivalence relation on it is called a setoid.

**Definition 43** (Partial Order). A (weak) partial order on a set  $X$  is reflexive, anti-symmetric and transitive binary relation on  $X$ .

**Definition 44** (Strict partial order). A strict partial order on a set  $X$  is anti-reflexive, asymmetric (and hence also anti-symmetric) and transitive relation on  $X$ .

**Definition 45** (Partially Ordered Set). A set  $X$  with a partial order relation  $R$  on it, denoted as  $(X, R)$  is a partially ordered set, or a poset.

**Definition 46** (Comparable Elements). Let  $X$  be a set partially ordered by  $R$ .  $a, b \in X$  are comparable if  $aRb \vee bRa$  is true, otherwise they are incomparable.

**Definition 47** (Interval on a Partially Ordered Set). Let  $X$  be a set partially ordered by relations  $\leq$  and  $<$ . The following intervals can be defined on  $X$  for all  $a, b \in X$ :

$$\begin{aligned} [a, b] &= \{x \in X : a \leq x \leq b\} & (a, b) &= \{x \in X : a < x < b\} \\ (a, b] &= \{x \in X : a < x \leq b\} & [a, b) &= \{x \in X : a \leq x < b\} \end{aligned}$$

**Definition 48** (Total Order, Linear Order, Chain). A total order (also, linear order, a chain) is a connex partial order – a binary relation that is reflexive, anti-symmetric, transitive, and connex.

**Definition 49** (Strict total order). A strict total order is a strict connex partial order.

**Definition 50** (Totally Ordered Set). A totally ordered (also, linearly ordered) set is a set equipped with a total order relation.

**Definition 51** (Minimal Element of a Poset). Let  $P$  be a set partially ordered by  $R$ , and let  $S \subseteq P$ . Element  $m \in S$  is a minimal element of  $S$  if  $\forall x \in S : xRm \implies x = m$ .

**Definition 52** (Maximal Element of a Poset). Let  $P$  be a set partially ordered by  $R$ , and let  $S \subseteq P$ . Element  $m \in S$  is a maximal element of  $S$  if  $\forall x \in S : mRx \implies x = m$ .

**Definition 53** (Least Element of a Poset). Let  $P$  be a set partially ordered by  $R$ , and let  $S \subseteq P$ . Element  $l \in S$  is the least element of  $S$  if  $\forall x \in S : lRx$ .

**Definition 54** (Greatest Element of a Poset). Let  $P$  be a set partially ordered by  $R$ , and let  $S \subseteq P$ . Element  $g \in S$  is the greatest element of  $S$  if  $\forall x \in S : xRg$ .

**Definition 55** (Minimum in a Totally Ordered Set). In case of a total order, the notions of minimal and least elements coincide, and such element is called minimum element.

**Definition 56** (Maximum in a Totally Ordered Set). In case of a total order, the notions of maximal and greatest elements coincide, and such element is called maximum element.

**Corollary 1.** A finite chain always has a greatest and a least element.

**Definition 57** (Upper Bound of a Partially Ordered Set). Let  $P$  be a set partially ordered by  $R$ . Let  $S \subseteq P$ . Element  $\lambda \in P$  is an upper bound of  $S$  if  $\forall x \in S : xR\lambda$ .

**Definition 58** (Lower Bound of a Partially Ordered Set). Let  $P$  be a set partially ordered by  $R$ . Let  $S \subseteq P$ . Element  $\lambda \in P$  is a lower bound of  $S$  if  $\forall x \in S : \lambda R x$ .

**Definition 59** (Supremum of a partially ordered set). Let  $P$  be a set partially ordered by  $R$ . Let  $S \subseteq P$ . An upper bound  $I$  of  $S$  is a supremum (denoted as  $\sup S$ ) if  $I$  is the least upper bound:  $\forall U \in P : I R U$ .

**Definition 60** (Infimum of a partially ordered set). Let  $P$  be a set partially ordered by  $R$ . Let  $S \subseteq P$ . A lower bound  $I$  of  $S$  is an infimum (denoted as  $\inf S$ ) if  $I$  is the greatest lower bound:  $\forall U \in P : U R I$ .