ITC8190
Mathematics for Computer Science
Equivalence Relations on Sets

Aleksandr Lenin

October 2nd, 2018

Relation R on a set A is called an **equivalence relation** iff R is reflexive, symmetric, and transitive.

Let us verify if $=$ is an equivalence relation on $\mathbb{N}$.

Reflexivity: any element a is equal to itself ($a = a$).
Symmetry: if $a = b$ then also $b = a$.
Transitivity: if $a = b$ and $b = c$, then also $a = c$.

Hence, $=$ is an equivalence relation on $\mathbb{N}$.

$$R \subseteq \mathbb{N} \times \mathbb{N} = \{(0,0), (1,1), (2,2), \ldots\} \ .$$

Suppose that f and g are differentiable functions on $\mathbb{R}$. Let $\sim$ be an equivalence relation defined by

$$f(x) \sim g(x) \Longleftrightarrow \frac{\partial f}{\partial x} = \frac{\partial g}{\partial x} \ .$$

It is clear that $\sim$ is reflexive and symmetric.

To show transitivity, suppose $f(x) \sim g(x)$ and $g(x) \sim h(x)$. The condition $\frac{\partial f}{\partial x} = \frac{\partial g}{\partial x}$ is satisfied if $f(x)$ and $g(x)$ differ by a constant.

$$\begin{aligned} f(x) - g(x) &= c_1 \ , \\ g(x) - h(x) &= c_2 \ , \\ f(x) - h(x) &= f(x) - g(x) + g(x) - h(x) = c_1 + c_2 \ . \end{aligned}$$

This implies $f(x) \sim h(x)$.

An equivalence relation gives rise to a partition via equivalence classes.

Picture will be drawn on the whiteboard.

Such a partition is called a **factor space**, and the following notation is used $X/\sim$, where X is the underlying set, and $\sim$ is the equivalence relation.

A set with an equivalence relation on it is called a **setoid**.

A **partition** P on a set X is a collection of non-empty subsets $X_1, X_2, \ldots$ such that are all disjoint, meaning that $X_i \cap X_j = \emptyset$ for $i \neq j$, and $\bigcup_k X_k = X$.

Let $\sim$ be an equivalence relation on a set X and let $x \in X$. Then the **equivalence class** $[x] \in X/\sim$ is

$$[x] = \{y \in X : y \sim x\} \ .$$

Lemma 1

Given an equivalence relation $\sim$ on a set X, there exists at least one non-empty equivalence class.

Proof.

Suppose there exists an equivalence relation $\sim$ on X, and let $x \in X$. By reflexivity of $\sim$, $x \sim x$, and so $x \in [x]$. Hence, the equivalence class $[x]$ is non-empty. $\square$

Theorem 1

Given an equivalence relation $\sim$ on a set X, the equivalence classes of X form a partition of X.

Proof.

Suppose there exists an equivalence relation $\sim$ on X. We need to show that $\sim$ forms a partition of X. By Lemma 1, $\bigcup_{x \in X} [x] = X$. Let $x, y \in X$. We will show that either $[x] \cap [y] = \emptyset$ or $[x] = [y]$. Suppose $[x] \cap [y]$ is non-empty.

$$z = [x] \cap [y] \neq \emptyset \implies z \sim x \wedge z \sim y \implies x \sim y \implies [x] \subseteq [y] \ .$$

Similarly, $y \sim x \implies [y] \subseteq [x]$, and so $[x] = [y]$. Therefore, two equivalence classes are disjoint or exactly the same. $\square$

Theorem 2

If $P = \{X_i\}$ is a partition of a set X_i, then there is an equivalence relation on X with equivalence classes X_i.

Proof.

Let $P = \{X_i\}$ be a partition of a set X. Let $a \sim b \iff a \in X_i \wedge b \in X_i$. Clearly, $\sim$ is reflexive. To show symmetry, observe that

$$x \sim y \implies x \in X_i \wedge y \in X_i \implies y \sim x \ .$$

For transitivity, observe that

$$x \sim y \wedge y \sim z \implies x \in X_i \wedge y \in X_i \wedge z \in X_i \implies x \sim z \ .$$

Clearly, $\sim$ is an equivalence relation on X. $\square$

Corollary 1

Any two equivalence classes are either disjoint or equal.

Corollary 2

Every equivalence relation on a set corresponds to a partition of this set.

Corollary 3

Any partition of a set corresponds to an equivalence relation which gives rise to this partition.

In example,

$$\mathbb{Z}/\sim: a \sim b \Longleftrightarrow a \equiv b \pmod 2$$

contains two equivalence classes [0] and [1] – even and odd numbers.

$$[0] = \{\ldots, -4, -2, 0, 2, 4, \ldots\}\ ,$$
$$[1] = \{\ldots, -3, -1, 1, 3, 5, \ldots\}\ .$$

It can be seen that $[0] \cap [1] = \emptyset$ and $[0] \cup [1] = \mathbb{Z}$.

Equivalence classes in $\mathbb{Z}/\sim: a \sim b \Longleftrightarrow a \equiv b \pmod 3$:

$$[0] = \{\ldots, -3, 0, 3, 6, \ldots\}\ ,$$
$$[1] = \{\ldots, -2, 1, 4, 7, \ldots\}\ ,$$
$$[2] = \{\ldots, -1, 2, 5, 8, \ldots\}\ ,$$

form another partition of $\mathbb{Z}$, since $[0] \cap [1] \cap [2] = \emptyset$ and $[0] \cup [1] \cup [2] = \mathbb{Z}$.

The set of integers $\mathbb{Z}$ is an image of $\mathbb{N} \times \mathbb{N}$, under $\sim$.

$$\mathbb{Z} = \mathbb{N} \times \mathbb{N} / \sim \ , \quad (a, b) \sim (c, d) \Longleftrightarrow a - b = c - d \ .$$

The set of rational numbers is an image of the set $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ under $\sim$.

$$\mathbb{Q} = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) / \sim \ , \quad (a, b) \sim (c, d) \Longleftrightarrow ac = bd \ .$$

The set $\mathbb{Z}_n$ is an image of $\mathbb{Z}$ under congruence relation.

$$\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\} = \mathbb{Z} / \equiv : a \equiv b \pmod{n} \Leftrightarrow n | (a - b) \ .$$

THANK YOU

FOR

YOUR

ATTENTION

ANY QUESTIONS?