ITC8190
Mathematics for Computer Science
Sets

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A **set** is a collection of objects defined in a manner that allows to determine for any given object x whether or not x belongs to the set.

$$X = \{x_1, x_2, \ldots, x_n\}$$
$$X = \{x : x \text{ satisfies } \mathcal{P}\}$$
$$\mathbb{N} = \{x : x \text{ is a natural number}\} = \{1, 2, 3, \ldots\}$$
$$\mathbb{Z} = \{x : x \text{ is an integer }\} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$
$$\mathbb{Q} = \{x : x \text{ is a rational number}\}$$
$$\mathbb{R} = \{x : x \text{ is a real number }\}$$
$$\mathbb{C} = \{x : x \text{ is a complex number}\}$$

Some examples:

The set of even numbers:

$$A = \{x \in \mathbb{Z} : 2 | x\} \ .$$

The set of odd numbers:

$$A = \{x \in \mathbb{Z} : 2 \not| x\} \ .$$

The set of prime numbers:

$$A = \{x \in \mathbb{Z} : \forall y \in \mathbb{Z}, y \neq 1, y \neq x : y \not| x\} \ .$$

The set of integers between 0 and 100 (inclusive):

$$A = \{x \in \mathbb{Z} : 0 \leqslant x \geqslant 100\} \ .$$

The set of integers that are multiples of 5:

$$A = \{x \in \mathbb{Z} : 5|x\} \ .$$

The set of complex numbers with absolute value 1.
The absolute value of $a + bi \in \mathbb{C}$ is $|a + bi| = \sqrt{a^2 + b^2}$.

$$A = \{a + bi \in \mathbb{C} : a^2 + b^2 = 1\} \ .$$

Set A is a **subset** of a set B (written as $A \subseteq B$) if membership in set A implies membership in set B.

$$A \subseteq B \iff a \in A \implies a \in B \ .$$

Sets A and B are **equal** if every set is a subset of the other.

$$A = B \iff A \subseteq B \land B \subseteq A \ .$$

Set A is a **proper subset** of a set B (written as $A \subset B$) if A is a subset of B, and A is not equal to B.

$$A \subset B \iff A \subseteq B \land A \neq B \ .$$

Some examples:

$$\{4, 5, 8\} \quad \subset \quad \{2, 3, 4, 5, 6, 7, 8, 9\}$$

$$\{4, 7, 9\} \quad \not\subseteq \quad \{2, 4, 5, 8, 9\}$$

$$\mathbb{N} \quad \subset \quad \mathbb{Z} \quad \subset \quad \mathbb{Q} \quad \subset \quad \mathbb{R} \quad \subset \quad \mathbb{C}$$

Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$. Show that A is a proper subset of B.

By definition of a proper subset:

$$A \subset B \Longleftrightarrow A \subseteq B \wedge A \neq B \ .$$

Indeed, $A \subseteq B$, since

$$\forall a \in A : a \in B \ .$$

To show that $A \neq B$, we show that there exists $5 \in B$, but $5 \notin A$, and so

$$B \not\subseteq A \implies B \neq A \ .$$

Therefore, $A \subset B$.

An **empty set** is a set for which

$$\forall x : x \notin \emptyset \ .$$

Union of sets A and B

$$A \cup B = \{x : x \in A \vee x \in B\} \ .$$

Intersection of sets A and B

$$A \cap B = \{x : x \in A \wedge x \in B\} \ .$$

The sets A and B are disjoint if

$$A \cap B = \emptyset.$$

Some examples:

$$\{1,2,3\} \cap \{4,5,6\} = \emptyset$$

$$A = \{x \in \mathbb{Z} : x > 2\} \quad B = \{x \in \mathbb{Z} : x \text{ is prime}\}$$
$$C = \{x \in \mathbb{Z} : x \text{ is even}\} \quad A \cap B \cap C = \emptyset$$

Let us show that the sets of even and odd numbers are disjoint.

By definition, the two sets are disjoint if their intersection is an empty set.

Let $A = \{x \in \mathbb{Z} : 2|x\}$ and $B = \{x \in \mathbb{Z} : 2 \not| x\}$.

We need to show that $A \cap B = \emptyset$.

$$\begin{aligned} A \cap B &\implies \{x \in A \wedge x \in B\} \\ &\implies \{x \in \mathbb{Z} \wedge 2|x \wedge 2 \not| x\} \\ &\implies \emptyset \ . \end{aligned}$$

Let

$$A = \{x \in \mathbb{Z} : 2|x\} \ ,$$
$$B = \{x \in \mathbb{Z} : 2 \not| x\} \ .$$

What is the set $A \cup B$?

$A \cup B = \mathbb{Z}$, since

$$\begin{aligned} A \cup B &\implies \{x \in A \vee x \in B\} \\ &\implies \{(x \in \mathbb{Z} \wedge 2|x) \vee (x \in \mathbb{Z} \wedge 2 \not| x)\} \\ &\implies \{x \in \mathbb{Z} \wedge (2|x \vee 2 \not| x)\} \\ &\implies \{x \in \mathbb{Z}\} = \mathbb{Z} \ . \end{aligned}$$

Let U be the universal set. The **complement of a set** A is the set

$$A' = \{x \in U : x \notin A\} \ .$$

The **difference** of the sets A and B is the set

$$A \setminus B = A \cap B' = \{x \in A : x \notin B\} \ .$$

Some examples:

$$\begin{aligned}
\{1,2,3\} \setminus \{4,5\} &= \{1,2,3\} \ . \\
\{1,2,3\} \setminus \{2,3,5\} &= \{1\} \ . \\
\{1,2,3,4\} \setminus \emptyset &= \{1,2,3,4\} \ . \\
\mathbb{Z} \setminus \{0\} &= \{\ldots,-2,-1,1,2,\ldots\} \ . \\
\mathbb{Z} \setminus \mathbb{N} &= \{\ldots,-3,-2,-1,0\} \ . \\
\mathbb{N} \setminus \mathbb{N} &= \emptyset \ .
\end{aligned}$$

The **Cartesian product** of sets A and B is the set of ordered pairs

$$A \times B = \{(a, b) : a \in A \wedge b \in B\} \ .$$

Let $A = \{x, y\}, B = \{1, 2, 3\}$. Then

$$A \times B = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\}$$
$$B \times A = \{(1, x), (2, x), (3, x), (1, y), (2, y), (3, y)\}$$

Observe that $A \times B \neq B \times A$.

The Cartesian product of a set with itself is often denoted by

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \ ,$$
$$\mathbb{Z}^n = \underbrace{\mathbb{Z} \times \ldots \times \mathbb{Z}}_{n \text{ times}} \ .$$

THANK YOU
FOR
YOUR
ATTENTION
ANY QUESTIONS?