ITC8190
Mathematics for Computer Science
Counting: Basic Methods

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Combinatorics is an area of mathematics concerned with

- existence
- construction
- optimization
- counting

of various finite structures.

Today we will focus on counting.

Obvious brute force solution:

- construct all instances, then count them one by one.

Tuple is a fixed-length ordered sequence of elements of a given m-element base set M.

Two tuples are considered the same if they have equal lengths and equal elements in corresponding positions.

An n-element tuple is called n-tuple.

For example, for the base set $M = \{1, 2, 3\}$, we have the following 9 distinct 2-tuples (pairs):
$(1, 1)$, $(1, 2)$, $(1, 3)$, $(2, 1)$, $(2, 2)$, $(2, 3)$, $(3, 1)$, $(3, 2)$, $(3, 3)$.

Permutation is a tuple with the additional requirement that all elements must be distinct.

For example, for the base set $M = \{1, 2, 3\}$, we have the following 6 distinct 2-permutations:
$(1, 2)$, $(1, 3)$, $(2, 1)$, $(2, 3)$, $(3, 1)$, $(3, 2)$.

An m-permutation of m-element base set M is called just a permutation of M.

Combination is a fixed-size subset of the base set.

Two combinations are considered the same if they are equal as sets, i.e. they consist of the same elements, irrespective of the order in which the elements are listed.

For example, for the base set $M = \{1, 2, 3\}$, we have the following 3 distinct 2-combinations:
$\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$.

Partition is a division of the base set M into a set of non-empty subsets in such a way that each element of M belongs to exactly one subset.

Two partitions are considered the same if they are equal as sets (of sets).

For example, for the base set $M = \{1, 2, 3\}$, we have the following 5 distinct partitions:
"partition" into one part: $\{\{1, 2, 3\}\}$,
into two parts: $\{\{1, 2\}, \{3\}\}$, $\{\{1, 3\}, \{2\}\}$, $\{\{2, 3\}, \{1\}\}$,
into three parts: $\{\{1\}, \{2\}, \{3\}\}$.

Rule of product: if there's w_1 ways of doing a first step and w_2 ways of doing a second step, then there are a total of $w = w_1 \cdot w_2$ ways of doing those two steps.

The total number of n-tuples of m-element set:

$$T(n, m) = \underbrace{m \cdot m \cdot \ldots \cdot m}_{n} = m^n.$$

The number of n-permutations of m-element set:

$$T(n, m) = m \cdot (m-1) \cdot \ldots \cdot (m-(n-1)) = \frac{m!}{(m-n)!} = (m)_n.$$

The rule of product can only be directly used if the number of possibilities for the second step is the same for all ways of doing the first step. This is a problem for applying the rule for counting n-combinations.

But, the rule of product can also be used in reverse: from each n-combination we can generate $n!$ distinct n-permutations, and this gives us each n-permutation exactly once; therefore the number of n-combinations:

$$C(n,m) = \frac{P(n,m)}{n!} = \frac{m!}{n! \cdot (m-n)!} = \binom{m}{n}.$$

Rule of sum: if there's w_1 ways of doing one step and w_2 ways of doing another step, and we must do one or the other, but not both, then there are a total of $w = w_1 + w_2$ ways of doing one of those two steps.

Recurrent formula for n-combinations of m-element set:

$$C(n,m) = \begin{cases} 1 & \text{if } n = 0 \\ 1 & \text{if } n = m \\ C(m-1, n-1) + C(m-1, n) & \text{if } 0 < n < m. \end{cases}$$

The rule of sum can be used twice to compute the number of partitions of an m-element set:

$$B(m) = \sum_{n=1}^{m} S(n, m),$$

where $S(n, m)$ is the number of partitions into n parts:

$$S(n, m) = \begin{cases} 1 & \text{if } n = 1 \\ 1 & \text{if } n = m \\ n \cdot S(n, m-1) + S(n-1, m-1) & \text{if } 1 < n < m. \end{cases}$$

Like the rule of product, the rule of sum can also be reversed.

For example, to count the integers in $1 \dots 100$ not divisible by 3, it is easier to count the number of integers that are divisible by 3 (which is 33) and then conclude that the remaining $100 - 33 = 67$ must be not divisible.

Inclusion-exclusion principle is a generalization of the rule of sum for overlapping sets.

For just two sets:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

For example, to count the integers in $1 \ldots 100$ divisible by 3 or 5 (or both), we need to observe that

- the number of those divisible by 3 is 33,
- the number of those divisible by 5 is 20,
- the number of those divisible by both 3 and 5, i.e. the number of those divisible by $\mathrm{lcm}(3, 5) = 15$ is 6,

and therefore the answer is $33 + 20 - 6 = 47$.

Inclusion-exclusion principle in general case:

$$|\bigcup_{i\in\{1,\ldots,n\}} A_i| = \Sigma_1 - \Sigma_2 + \ldots + (-1)^{n+1}\Sigma_n,$$

where

$$\Sigma_i = \sum_{\{j_1,\ldots,j_i\}\subseteq\{1,\ldots,n\}} |A_{j_1} \cap \ldots \cap A_{j_i}|.$$

This is quite inconvenient to use manually for larger n, though, as the number of summands in Σ_i is $\binom{n}{i}$ and the total number of summands over all Σ_i is 2^n.

THANK YOU
FOR
YOUR
ATTENTION
ANY QUESTIONS?