ITC8190
Mathematics for Computer Science
Recap and Preparation for the Test

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October 16th, 2018

Set Theory

Show that $(A \cap B)' = A' \cup B'$.

$$\begin{aligned}(A \cap B)' &= \{x : x \notin A \cap B\} \\ &= \{x : \neg(x \in A \wedge x \in B)\} \\ &= \{x : x \notin A \vee x \notin B\} \\ &= \{x : x \in A' \vee x \in B'\} \\ &= A' \cup B' \ .\end{aligned}$$

Set Theory

Show that $(A \cap B)' = A' \cup B'$.

$$
\begin{aligned}
x \in (A \cap B)' &\implies x \notin A \cap B \\
&\implies x \notin A \vee x \notin B \\
&\implies x \in A' \vee x \in B' \\
&\implies x \in A' \cup B' \\
&= (A \cap B)' \subseteq A' \cup B' \quad .
\end{aligned}
$$

$$
\begin{aligned}
x \in A' \cup B' &\implies x \notin A \vee x \notin B \\
&\implies x \notin A \cap B \\
&\implies x \in (A \cap B)' \\
&\implies A' \cup B' \subseteq (A \cap B)' \quad .
\end{aligned}
$$

Therefore, $(A \cap B)' = A' \cup B'$.

Partitions and Factor Spaces

A partition P of a set X is the set $P = \{X_1, X_2, \ldots, X_n\}$ such that

$$X_i \cap X_j = \emptyset \text{ for } i \neq j$$
$$\bigcup_i X_i = X.$$

Factor space is an image of a set under an equivalence relation, together with some binary operation on the set of equivalence classes.

$$\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$$
$$\mathbb{Z}_n = \mathbb{Z}/\equiv$$
$$a \equiv b \iff n|(a-b) \quad .$$

Factor space $\mathbb{Z}_n$ is a collection of equivalence classes

$$\mathbb{Z}_n = \{[0], [1], [2], \ldots, [n-1]\} \quad .$$

Partitions and Factor Spaces

In example, the left cosets of $H = \{0, 3\}$ in $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ are subsets $X_1 = \{0, 3\}$, $X_2 = \{1, 4\}$, $X_3 = \{2, 5\}$ that form a partition on $\mathbb{Z}_6$. It can be seen that

$$\begin{aligned} X_1 \cup X_2 \cup X_3 &= \mathbb{Z}_6 \ , \\ X_1 \cap X_2 &= \emptyset \ , \\ X_2 \cap X_3 &= \emptyset \ , \\ X_1 \cap X_3 &= \emptyset \ . \end{aligned}$$

Partitions and Factor Spaces

$$\mathbb{Z}_3 = \{[0],[1],[2]\} = \{0,1,2\} \ ,$$
$$[0] = \{\ldots,-6,-4,-2,0,2,4,6,\ldots\}$$
$$[1] = \{\ldots,-5,-3,-1,1,3,5,7,\ldots\}$$
$$[2] = \{\ldots,-6,-2,0,2,4,6,8,\ldots\}$$

It can be seen that $[0] \cap [1] = [0] \cap [2] = [1] \cap [2] = \emptyset$ and $[0] \cup [1] \cup [2] = \mathbb{Z}$. Therefore, $\mathbb{Z}_3$ partitions $\mathbb{Z}$ into 3 equivalence classes $[0],[1],[2]$. Similarly,

$$\mathbb{Z}_4 = \{0,1,2,3\} \ , \quad \mathbb{Z}_5 = \{0,1,2,3,4\} \ , \quad \mathbb{Z}_6 = \{0,1,2,3,4,5\} \ .$$

Cartesian Products

$$\mathbb{Z}_2^3 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),\\ (1,0,0),(1,0,1),(1,1,0),(1,1,1)\} \ .$$

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0),(0,1),(1,0),(1,1)\} \ .$$

$$\mathbb{Z}_2 \times \mathbb{Z}_3 = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2)\} \ .$$

$$\mathbb{Z}_3 \times \mathbb{Z}_2 = \{(0,0),(0,1),(1,0),(1,1),(2,0),(2,1)\} \ .$$

Cartesian Products

$$\begin{aligned}\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_2 = \{&(0,0,0),(0,0,1),(0,1,0),(0,1,1),\\ &(0,2,0),(0,2,1),(1,0,0),(1,0,1),\\ &(1,1,0),(1,1,1),(1,2,0),(1,2,1)\}\ .\end{aligned}$$

$$\begin{aligned}\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \{&(0,0,0),(0,0,1),(0,1,0),(0,1,1),\\ &(1,0,0),(1,0,1),(1,1,0),(1,1,1),\\ &(2,0,0),(2,0,1),(2,1,0),(2,1,1)\}\ .\end{aligned}$$

Binary Relations

Show that function $\phi : \mathbb{Z} \to 2\mathbb{Z}$ is injective.

$$2n = 2m \implies n = m \ .$$

Show that function $\phi : \mathbb{Z} \to \mathbb{Z}$ defined by $\phi : n \mapsto n^2$ is not injective. It can be seen that $a^2 = (-a)^2$, but $a \neq -a$.

$$a^2 = b^2 \not\Longrightarrow a = b \ .$$

Show that function $\phi : \mathbb{Z} \to \mathbb{Z}$ defined by $\phi : x \mapsto x + 10$ is surjective. It can be seen that for every integer $z \in \mathbb{Z}$ there exists its unique preimage $z' = z - 10 \in \mathbb{Z}$, such that $z - 10 + 10 = z$.

$$\forall z \in \mathbb{Z} \ \exists z' = z - 10 : z = \phi(z') \ .$$

Bijections

Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. Define a mapping $\phi : A \to B$ by

$$\phi : 1 \mapsto a \ , \qquad 2 \mapsto b \ , \qquad 3 \mapsto c \ .$$

The mapping $\phi : A \to B$ is a bijection iff it is invertible. Define an inverse mapping $\psi : B \to A$ by

$$\psi : a \mapsto 1 \ , \qquad b \mapsto 2 \ , \qquad c \mapsto 3 \ .$$

Bijections

It must hold that

$$(\psi \circ \phi)(a) = a \ ,$$
$$(\phi \circ \psi)(b) = b \ .$$

The compositions are:

$$\psi \circ \phi : A \to A = id_A \ , \phi \circ \psi : B \to B = id_B \ .$$

It can be seen that

$$\begin{aligned} (\psi \circ \phi)(1) &= \psi(a) = 1 \qquad & (\phi \circ \psi)(a) &= \phi(1) = a \\ (\psi \circ \phi)(2) &= \psi(b) = 2 \qquad & (\phi \circ \psi)(b) &= \phi(2) = b \\ (\psi \circ \phi)(3) &= \psi(c) = 3 \qquad & (\phi \circ \psi)(c) &= \phi(3) = c \end{aligned}$$

Bijections

Consider a mapping $\phi : \mathbb{Z} \to 3\mathbb{Z}$ given by $\phi : n \mapsto 3n$. To show that $\phi : \mathbb{Z} \to 3\mathbb{Z}$ is a bijection, consider an inverse mapping $\psi : 3\mathbb{Z} \to \mathbb{Z}$ by $\psi : n \mapsto \frac{n}{3}$. Then

$$(\phi \circ \psi)(a) = \phi\left(\frac{a}{3}\right) = 3 \cdot \frac{a}{3} = a \ ,$$
$$(\psi \circ \phi)(a) = \psi(3a) = 3a \cdot \frac{1}{3} = a \ .$$

Therefore, $\phi : \mathbb{Z} \to 3\mathbb{Z}$ is a bijection.

Bijections

Consider a mapping $\phi : \mathbb{Z} \to \mathbb{Z}$ defined by $\phi : x \mapsto x + 15$. To show that $\phi : \mathbb{Z} \to \mathbb{Z}$ is a bijection, define an inverse mapping $\psi : \mathbb{Z} \to \mathbb{Z}$ by $\psi : x \mapsto x - 15$. Then

$$(\phi \circ \psi)(a) = \phi(a - 15) = a - 15 + 15 = a \ ,$$
$$(\psi \circ \phi)(a) = \psi(a + 15) = a + 15 - 15 = a \ .$$

Therefore, $\phi : \mathbb{Z} \to \mathbb{Z}$ is a bijection.

Composition of Mappings

Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f : n \mapsto n + 5$, and $g : \mathbb{Z} \to 2\mathbb{Z}$ be defined by $g : n \mapsto 2n$. Then

$$(f \circ g)(x) = f(2x) = 2x + 5 \ ,$$
$$(g \circ f)(x) = g(x + 5) = 2(x + 5) = 2x + 10 \ .$$

The inverse mappings $f^{-1} : \mathbb{Z} \to \mathbb{Z}$ defined by $f : n \mapsto n - 5$ and $g^{-1} : 2\mathbb{Z} \to \mathbb{Z}$ defined by $g : n \to \frac{n}{2}$.

$$((f \circ g)^{-1})(x) = (g^{-1} \circ f^{-1})(x) = g^{-1}(x - 5) = \frac{x - 5}{2} \ ,$$
$$((f \circ g) \circ (g^{-1} \circ f^{-1}))(x) = \frac{(2x + 5) - 5}{2} = \frac{2x}{2} = x \ .$$

Equivalence Relation

Show that group isomorphism $\cong$ is an equivalence relation on the class of groups. Groups $(G, \odot)$ and $(H, \circ)$ are said to be **isomorphic** (written $G \cong H$) iff there exists a bijection $\phi : G \to H$ that preserves group operations.

$$\forall a, b \in G : \phi(a \odot b) = \phi(a) \circ \phi(b) \quad .$$

Reflexivity: $G \cong G$

Symmetry: $G \cong H \implies H \cong G$

Transitivity: $G \cong H \cong K \implies G \cong K$

Partial order relation

Show that $|$ is a partial order relation on the set A. Let $a, b, c \in A$.

Reflexivity: $a|a$

Anti-symmetry: $a|b \wedge b|a \implies a = b$

Transitivity: $a|b \wedge b|c \implies a|c$

Show that $<$ is a strict partial order relation on the set A.

Anti-reflexivity: $a \not< a$

Asymmetry: $a < b \implies \neg(b < a)$

Transitivity: $a < b < c \implies a < c$

THANK YOU
FOR
YOUR
ATTENTION
ANY QUESTIONS?