Exercise 1. Let p, q, r, s, t, u be integers, where q, s, u are non-zero. A relation R is defined by

$$\frac{p}{q} \sim \frac{r}{s} \Longleftrightarrow ps = qr \ .$$

Show that R is an equivalence relation.

Exercise 2. For (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$, relation R is defined by

$$(x_1, y_1) \sim (x_2, y_2) \Longleftrightarrow x_1^2 + y_1^2 = x_2^2 + y_2^2 \ .$$

Show that R is an equivalence relation.

Exercise 3. Determine whether or not the following relations are equivalence relations on the given set. Show which properties of an equivalence relations hold and which not.

(a) $x \sim y$ in $\mathbb{R}$ if $x \geqslant y$,

(b) $m \sim n$ in $\mathbb{Z}$ if $mn > 0$,

(c) $x \sim y$ in $\mathbb{R}$ if $|x - y| \leqslant 4$,

(d) $m \sim n$ in $\mathbb{Z}$ if $m \equiv n \pmod 6$.

Exercise 4. Define a relation $\sim$ on $\mathbb{R}^2$ by stating that

$$(a, b) \sim (c, d) \Longleftrightarrow a^2 + b^2 \leqslant c^2 + d^2 \ .$$

Show that $\sim$ is reflexive, transitive, but not symmetric.

Exercise 5 (Projective Real Line $\mathbb{P}(\mathbb{R})$)**.** Define a relation on $\mathbb{R}^2 \setminus (0, 0)$:

$$(x_1, y_1) \sim (x_2, y_2) \Longleftrightarrow \exists \lambda \in \mathbb{R}, \lambda \neq 0 : (x_1, y_1) = (\lambda x_2, \lambda y_2) \ .$$

Show that $\sim$ defines an equivalence relation on $\mathbb{R}^2 \setminus (0, 0)$.

Exercise 6. Let $\mathbb{Z}*$ be the set of all non-zero integers, and let R be a relation on $\mathbb{Z} \times \mathbb{Z}*$ given by

$$\forall x, y \in \mathbb{Z}, \forall x', y' \in \mathbb{Z}* : (x, y) R (x', y') \Longleftrightarrow xy' = x'y \ .$$

Show that R is an equivalence relation.