Hybrid Systems, Lecture 6: Stability of Hybrid Systems

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Hybrid Automaton

- Let $H = (Q, X, Init, f, D, G, R, E)$;
- $Q = q_1, \ldots, q_k$ - is a finite set of discrete states (control locations);
- $X = (x_1, \ldots x_n)$ - is a finite set of continuous variables;
- $f : Q \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ - is an activity function;
- $Init \subset Q \times \mathbb{R}^n$ - is the set of initial states;
- $D : Q \rightarrow 2\mathbb{R}^n$ - invariants of the locations (domains);
- $E \subseteq Q \times Q$ - is the transition relation;
- $G : E \rightarrow 2^{\mathbb{R}^n}$ - is is the guard condition;
- $R : E \rightarrow 2^{\mathbb{R}^n} \times 2^{\mathbb{R}^n}$ - is the reset map;

Solution of Hybrid Automaton

- $\mathcal{X} = (\tau, q, x)$
- *Initialization* $(q(0), x^0(0)) \in \text{Init}$;
- *Time driven* $\forall t \in [\tau_i, \tau_i'), \quad \dot{x}^i(t) = f(q(i), x^i(t))$ and $x^i(t) \in D(q(i))$
- *Event driven* $\forall i \in \langle\tau\rangle \setminus N, e = (q(i), q(i+1)) \in E$, $x^i(\tau_i') \in G(e)$ and $x^{i+1}(\tau_{i+1}) \in R(e, x^i(\tau_i'))$

Switched systems

Let $\Omega_q, q = 1, \ldots, m$ denote a partition of the continuous state space $\mathbb{R}^n$.
A *switched system* is then defined as

$$\dot{x} = f_q(x), \quad x \in \Omega_q$$

Consider following example:

$$x \in \mathbb{R}^2,$$

Ω_q - is a partition where q - is a quadrant $q = 1, \ldots, 4$,

$$\begin{aligned} \dot{x} &= A_q x \\ x &\in \Omega_q \end{aligned}$$

Stability

A solution x^* of a switched system is stable if for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that for all solutions x

$$\|x(0) - x^*(0)\| < \delta \Rightarrow \|x(t) - x^*(t)\| < \epsilon, \forall t > 0$$

Lyapunov's Second Method

Let $x^* = 0$ be an equilibrium point of $\dot{x} = f(x)$. If there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$\begin{aligned} V(0) &= 0 \\ V(x) &> 0, \quad \forall x \in \mathbb{R}^n \{0\} \\ \dot{V}(x) &\leq 0, \quad \forall x \in \mathbb{R}^n \end{aligned}$$

then x^* is stable.

Lyapunov Function for Linear System

A Lyapunov function for a linear system

$$\dot{x} = Ax$$

is given by

$$\begin{aligned} V(x) &= x^T P x \\ \dot{V}(x) &= -x^T Q x < 0 \end{aligned}$$

Example

$$\dot{x} = A_1 x = \begin{pmatrix} -1 & 10 \\ -100 & -1 \end{pmatrix} x$$

Then

$$P = \begin{pmatrix} 0.2752 & -0.0225 \\ -0.0225 & 2.7478 \end{pmatrix}$$

Solution of the Lyapunov equation $A_1 P + P A_1^T = -I$. Leads $V = x^T P x$ is stable. (fulfills the conditions of the Lyapunov theorem).
find $\lambda(A_1)$

Stable + Stable =Unstable

Consider the following switched system:

$$v_1: \begin{array}{rcl} \dot{x} & = & A_1 x \\ x_1 x_2 & \leq & 0 \end{array}; \qquad v_2: \begin{array}{rcl} \dot{x} & = & A_2 x \\ x_1 x_2 & \geq & 0 \end{array}$$

$$(q_1, q2) = (x_1 x_2 \geq 0)$$

$$(q_2, q1) = (x_1 x_2 \leq 0)$$

$$A_1 = \begin{pmatrix} -1 & 10 \\ -100 & -1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -1 & 100 \\ -10 & -1 \end{pmatrix}$$

Is this system defined correctly?
What should be changed to make it stable?

Multiple Lyapunov Functions

Let us suppose $x^* = 0$ is an equilibrium of each mode $= 1, \ldots, m$ of the switched system

$$\dot{x} = f_q(x), \quad x \in \Omega_q$$

If there exist function $V_1, \ldots V_m$ such that

$$\begin{aligned} V_q(0) = 0, \quad V_q(x) > 0, \quad \forall x \in \mathbb{R}^n \ \{0\} \\ \dot{V}_q\big((x(t))\big) \qquad \leq 0, \qquad \forall x(t) \in \Omega_q \end{aligned}$$

and the sequences $\left\{ V_q\big(x(\tau_{i_q})\big) \right\}, \quad q = 1, \ldots, m$ are non-increasing, where τ_{i_q} are the time instances when model q becomes active, then x^* is stable.

Supervisory Control

- The goal is to choose switching $\sigma = \sigma(t)$ such that $\dot{x} = f_\sigma(x)$ possess desired property.
- Supervisory control: *supervisor* decide which controller is active.
  Switching signal $\sigma : [0, \infty) \rightarrow \{1, \ldots, m\}$
- Arbitrary switching:In some cases σ may be chosen arbitrary and still stabilize the system.

Common Lyapunov Function

Consider the system

$$\dot{x} = A_\sigma x$$

where $\sigma : [0, \infty) \to \{1, \ldots m\}$ is an arbitrary switching sequence. If there exists P, Q_q¿0 such that

$$PA_q + A_q^T P = -Q_q, \quad q = 1, \ldots, m$$

then the origin is stable.
$V(x) = x^T P x$ is a common Lyapunov function for all the systems $\dot{x} = A_q x$

A Stabilizing Switching Sequence

- Consider the system $\dot{x} = A_\sigma x$, where $\sigma : [0, \infty) \to \{1, \ldots m\}$ is an arbitrary switching sequence. If all A_σ are stable and $A_k A_l = A_l A_k \;\; k, l \in \{1, \ldots, m\}$ then the origin is stable.
- Suppose there exist $\mu_q \geq 0$, $q \in Q$ and $\sum_{q=1}^{m} \mu_k = 1$, such that $A = \sum_{q=1}^{m} \mu_k A_k$ is stable. Then, a stabilizing switching sequence $\sigma : [0, \infty) \to Q = \{1, \ldots m\}$ for

$$\dot{x} = A_\sigma x$$

is given by

$$\sigma(x) = \arg\min_{q \in Q} x^T \left(A_q^T P + P A_q\right) x$$

where $P > 0$ is the solution of $A^T P + PA = -I$

Implementation

- Xcos
- pure script