Lecture 7:
Introduction to formal specifications

Lecture notes by Mike Gordon are used

24.03.2016

Recall some definitions

- *Formal Specification* - **using mathematical notation to give a precise description of what a program should do**

- *Formal Verification* - **using precise rules to mathematically prove that a program satisfies a formal specification**

- *Formal Development (Refinement)* - **developing programs in a way that ensures mathematically they meet their formal specifications**

Introduction

- Verification of programs is based on formal specification and on related verification method.

  We will use Floyd-Hoare logic (FHL)

- Proof systems of the FHL style depend on particular programming language with its syntax and semantics

- In this course we will deal with the verification of

  - deterministic sequential *while*-programs;
  - non-deterministic sequential *while*-programs
  - parallel programs with shared variables;
  - parallel programs with message passing.

Programs as state transition systems

- Programs are structured specifications of state transition systems.
- Programming language defines constructs for specifying single transitions and transition compositions.
- State components specified using datatypes are referred in conditions of command constructs like *if*-, *while*-, *for*-, *case*-command etc.

Some notations

- Programs are built out of *commands* like assignment, *if*-, *while*-, *for*-, *case*-command etc
- The terms '*program*' and '*command*' are synonymous.
- '*Program*' will only be used for commands representing <u>complete algorithm</u>.
- The '*statement*' and '*assertion*' are used for conditions on program variables that occur in correctness specifications.

Imperative programs - state

- Executing an imperative program has the effect of changing the *state*
  - i.e. the values of program variables
  - N.B. languages are more complex than those described in our course
  - they may have states consisting of other things than the values of variables (e.g. I/O ports).

Imperative programs - execution

- To use an imperative program
  - first establish a state,
    i.e. set some variables to have values of interest
  - then execute the program,
    (to transform the initial state into a final one)
  - inspect the values of variables in the final state to get the result.

Simple *while*-language

% Expressions

- E ::= N|V|$E1$+$E2$|$E1$-$E2$|$E1\times E2$| … % Arithmetic
- B ::= T|F|$E1$=$E2$|$E1\leq E2$| … % Logic

- C ::= %Commands:

```
      SKIP                                  % empty command (place holder)
    | V := E                                % assignment
    | V(E1) := E2                           % array assignment
    | C1 ; C2                               % sequential execution
    | IF B THEN C1 ELSE C2                  % conditional execution
    | BEGIN VAR V1;…;VAR Vn; C END          % block command (var. scoping)
    | WHILE B DO  C                         % while - loop
    | FOR V := E1 UNTIL E2  DO C            % for - loop
```

Terminology and notations

- ***Variable***
  - *V*1, *V*2, ..., *Vn*
- ***Program state*** - valuation of program (and control) variables
- ***Command*** - gives a rule how the program state changes
  - *C*1, *C*2, ... , *Cn*
- ***Program*** - command that includes all the commands in the algorithm
  - *C*
- ***Expression***
  - Arithmetic expression gives a value: *E*1, *E*2, ... , *En*
  - Boolean expression gives a *truth*-value: *B*1, *B*2, ... , *Bn*
- ***Statement*** – logical expression on program variables in the pre- and postconditions of the specification
  - *S*1, *S*2, ... , *Sn*

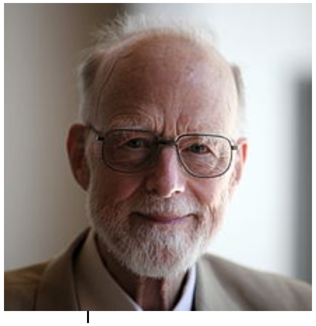
Formal specification

- Describes the intended behaviour of the program
- Specifies *what* the program must do
- Has well-defined ***synax*** and ***semantics***
- that helps avoiding ***ambiguous*** and ***controversial*** specifications
- Can be used to prove the ***correctness of the program***
- Can be used to generate ***tests*** and ***counterexamples***

We will use formalism that is based on FHL and predicate calculus

Hoare's notation

Sir Tony Hoare



- **C.A.R. Hoare introduced the following notation called a *partial correctness specification* for specifying what a program does:**

$$\{P\}\ C\ \{Q\}$$

where:

- C **is a program from the programming language whose programs are being specified**
- P **and** Q **are conditions on the program variables used in** C

Hoare's notation

- **Conditions on program variables will be written using standard mathematical notations together with** ***logical operators*** **like:**
  - $\wedge$ **('and'),** $\vee$ **('or'),** $\neg$ **('not'),** $\Rightarrow$ **('implies')**

- **Hoare's original notation was** $P\ \{C\}\ Q$ **not** $\{P\}\ C\ \{Q\}$**, but the latter form is now more widely used**

Partial Correctness

- **An expression** $\{P\}\ C\ \{Q\}$ **is called a** *partial correctness specification*
  - P **is called its** *precondition*
  - Q **its** *postcondition*
- $\{P\}\ C\ \{Q\}$ **is true if**
  - **whenever** C **is executed in a state satisfying** P
  - **and** *if* **the execution of** C **terminates**
  - **then the state in which** C**'s execution terminates satisfies** Q

Examples

- $\{\mathtt{X} = 1\}\ \mathtt{Y:=X}\ \{\mathtt{Y} = 1\}$
  - **This says that *if* the command `Y:=X` is executed in a state satisfying the condition $\mathtt{X} = 1$**
  - **i.e. a state in which the value of `X` is 1**
  - ***then*, if the execution terminates (which it does)**
  - **then the condition $\mathtt{Y} = 1$ will hold**
  - **Clearly this specification is true**

Examples

- $\{X = 1\}$ `Y:=X` $\{Y = 2\}$
  - This says that if the execution of `Y:=X` terminates when started in a state satisfying $X = 1$
  - then $Y = 2$ will hold
  - This is clearly false
- $\{X = 1\}$ `WHILE T DO SKIP` $\{Y = 2\}$
  - This specification is true!

Total correctness

- **A stronger kind of specification is a** *total correctness specification*
  - **There is no standard notation for such specifications**
  - **We shall use** $[P]\ C\ [Q]$
- **A total correctness specification** $[P]\ C\ [Q]$ **is true if and only if**
  - **Whenever** C **is executed in a state satisfying** P**, then the execution of** C **terminates**
  - **After** C **terminates** Q **holds**

Example

- $[\texttt{X} = 1]$ `Y:=X; WHILE T DO SKIP` $[\texttt{Y} = 1]$
  - **This says that the execution of** `Y:=X;WHILE T DO SKIP` **terminates when started in a state satisfying** $\texttt{X} = 1$
  - **after which** $\texttt{Y} = 1$ **will hold**
  - **This is clearly false**

Total correctness

- **Informally:**

$$\begin{aligned} &\mathit{Total\ correctness} = \\ &\quad \mathit{Termination} + \mathit{Partial\ correctness} \end{aligned}$$

- **Total correctness is the ultimate goal**
  - **usually easier to show partial correctness and termination separately**

Total correctness

- **Termination is usually straightforward to show, but there are examples where it is not: no one knows whether the program below terminates for all values of X**

```
WHILE X>1 DO
  IF ODD(X) THEN X := (3×X)+1 ELSE X := X DIV 2
```

  - **The expression X DIV 2 evaluates to the result of rounding down X/2 to a whole number**

- **Exercise: Write a specification which is true if and only if the program above terminates**

Auxiliary variables in the specification

- $\{X{=}x \land Y{=}y\}$ `R:=X; X:=Y; Y:=R` $\{X{=}y \land Y{=}x\}$
  - **This says that *if* the execution of**

    `R:=X; X:=Y; Y:=R`

    terminates (which it does)
  - ***then* the values of `X` and `Y` are exchanged**
- **The variables `x` and `y`, which don't occur in the command and are used to name the initial values of program variables `X` and `Y`**
- **They are called *auxiliary* variables**

Examples

- $\{X{=}x \wedge Y{=}y\}$ `BEGIN X:=Y; Y:=X END` $\{X{=}y \wedge Y{=}x\}$
  - **This says that** `BEGIN X:=Y; Y:=X END` **exchanges the values of** `X` **and** `Y`
  - **This is not true**

Examples

- $\{\mathtt{T}\}\ C\ \{Q\}$
  - **This says that whenever C halts, Q holds**
- $\{P\}\ C\ \{\mathtt{T}\}$
  - **This specification is true for every condition P and every command C**
  - **Because $\mathtt{T}$ is always true**

Examples

- $[P]\ C\ [\mathtt{T}]$
  - **This says that C terminates if initially P holds**
  - **It says nothing about the final state**
- $[\mathtt{T}]\ C\ [P]$
  - **This says that C always terminates and ends in a state where P holds**

A more complicated example

```
{T}
  BEGIN
    R:=X;
    Q:=0;
    WHILE Y≤R DO
      BEGIN R:=R-Y; Q:=Q+1 END
  END
{R < Y ∧ X = R + (Y × Q)}
```

C (the braced command from BEGIN to END)

- **This is** $\{\mathtt{T}\}\ C\ \{\mathtt{R} < \mathtt{Y}\ \wedge\ \mathtt{X} = \mathtt{R} + (\mathtt{Y} \times \mathtt{Q})\}$
  - **where** C **is the command indicated by the braces above**
  - **The specification is true if whenever the execution of** C **halts, then** `Q` **is quotient and** `R` **is the remainder resulting from dividing** `Y` **into** `X`
  - **It is true (even if** `X` **is initially negative!)**
  - **In this example a program variable** `Q` **is used. This should not be confused with the** Q **used in previous examples to range over postconditions**

Some exercises

- **When is $[\mathtt{T}]\ C\ [\mathtt{T}]$ true?**
- **Write a partial correctness specification which is true if and only if the command C has the effect of multiplying the values of $\mathtt{X}$ and $\mathtt{Y}$ and storing the result in $\mathtt{X}$**
- **Write a specification which is true if the execution of C always halts when execution is started in a state satisfying P**

Specification can be tricky : Sorting

- **Suppose $\mathtt{C}_{sort}$ is a command that is intended to sort the first n elements of an array**

- **To specify this formally, let $\mathtt{SORTED}(A, n)$ mean**

$$A(1) \leq A(2) \leq \ldots \leq A(n)$$

Sorting: naive spec

- **A first attempt to specify that $\mathtt{C}_{sort}$ sorts is**

$$\{1 \le \mathtt{N}\}\ \mathtt{C}_{sort}\ \{\mathtt{SORTED(A,N)}\}$$

- **Not enough:**
  - $\mathtt{SORTED(A,N)}$ **can be achieved by simply zeroing the first $\mathtt{N}$ elements of $\mathtt{A}$**

Sorting: permutation required

- **It is necessary to require that the sorted array is a rearrangement, or permutation, of the original array**

- **To formalise this, let** $\texttt{PERM}(A, A', N)$ **mean that**

  $$A(1), A(2), \ldots, A(n)$$

  is a rearrangement of

  $$A'(1), A'(2), \ldots, A'(n)$$

- **An improved specification that** $\texttt{C}_{sort}$ **sorts:**

$$\{\texttt{1}\leq\texttt{N} \wedge \texttt{A=a}\}\ \texttt{C}_{sort}\ \{\texttt{SORTED(A,N)} \wedge \texttt{PERM(A,a,N)}\}$$

Sorting: still not correct

- **The following specification is true**

  $\{1 \leq \mathtt{N}\}$

  $\mathtt{N:=1}$

  $\{\mathtt{SORTED(A,N)} \wedge \mathtt{PERM(A,a,N)}\}$

- **Must say explicitly that `N` is unchanged**

Sorting: still not correct

- **A better specification is thus:**

$$\{1 \leq \mathtt{N} \wedge \mathtt{A=a} \wedge \mathtt{N=n}\}$$
$$\mathtt{C}_{sort}$$
$$\{\mathtt{SORTED(A,N)} \wedge \mathtt{PERM(A,a,N)} \wedge \mathtt{N=n}\}$$

- **Is this the correct specification?**
  - **What if `N` is larger than the size of the array?**

Summary

- **We have given a notation for specifying**
  - partial correctness of programs
  - total correctness of programs
- **It is easy to write incorrect specifications**
  - and we can prove the correctness of the incorrect programs
- **It is recommended to use testing, simulation and formal verification hand in hand.**