ITC8190
Mathematics for Computer Science
Order Relations on Sets

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partial order is reflexive, anti–symmetric and transitive binary relation on a set.

I.e. the relation $\leqslant$ is a partial order on $\mathbb{Z}$.

strict partial order is anti–reflexive partial order. It is a binary relation which is anti–reflexive, anti–symmetric (and hence also asymmetric), and transitive.

I.e. the relation $<$ is a strict partial order on $\mathbb{Z}$.

A set with a partial order on it is called a **partially ordered set** or a **poset**.

I.e. $(\mathbb{Z}, \leqslant)$ is a poset. $(\mathcal{P}(\{a, b, c\}), \subseteq)$ is a poset.

For elements a, b of a partially ordered set P, if $a \leqslant b$ or $b \leqslant a$, then a and b are **comparable**. Otherwise they are **incomparable**.

Let us show that the relation $\leqslant$ is a partial order on some set A. It can be seen that for all $a, b, c \in A$, it holds that

$$
\begin{aligned}
& a \leqslant a \;\; \text{(reflexivity)} \quad , \\
& a \leqslant b \wedge b \leqslant a \implies a = b \;\; \text{(anti–symmetry)} \quad , \\
& a \leqslant b \wedge b \leqslant c \implies a \leqslant c \;\; \text{(transitivity)} \quad .
\end{aligned}
$$

Similarly, $<$ is a strict partial order on A.

$$
\begin{aligned}
& \neg(a < a) \;\; \text{(anti–reflexivity)} \quad , \\
& a < b \implies \neg(b < a) \;\; \text{(asymmetry)} \quad , \\
& \underbrace{a < b \wedge b < a}_{\text{always false}} \implies a = b \text{ anti–symmetry} \; , \\
& a < b \wedge b < c \implies a < c \;\; \text{(transitivity)} \quad .
\end{aligned}
$$

For a set X with a partial order relation $\leqslant$ on it, the interval $[a, b]$ is the set

$$[a, b] = \{x \in X : a \leqslant x \leqslant b\} \ .$$

Using the corresponding strict relation $<$, an open interval (a, b) on X is the set

$$(a, b) = \{x \in X : a < x < b\} \ .$$

Half–open intervals $[a, b)$ and $(a, b]$ are defined similarly.

Let us show that divisibility ($a < b \iff a|b$) is a partial order relation on a set A.

Reflexivity: $\forall x \in A : x|x$.
Anti–symmetry: $\forall x, y \in A : x|y \wedge y|x \implies x = y$.
Transitivity: $\forall x, y, z \in A : x|y \wedge y|z \implies x|z$.

$$\exists \alpha, r \in A : y = \alpha x + r \ ,$$
$$\exists \beta, s \in A : z = \beta y + s \ ,$$
$$z = \beta(\alpha x + r) + s = (\beta\alpha)x + (\beta r + s) \implies x|z \ .$$

Definition 1 (Greatest Common Divisor)

Let A be a set. Let $a, b \in A$. The greatest common divisor of a and b is an element $d = \gcd(a, b)$ such that $d|a$, $d|b$, and any other common divisor c of a and b, divides d.

$$\forall c \in A : c|a \wedge c|b \implies c|d \ .$$

W.r.t. this definition

$$\gcd(4, 6) = 2 \ ,$$
$$\gcd(4, 6) = -2 \ .$$

This definition of gcd is general, and allows us to calculate the gcd of two polynomials. I.e.:

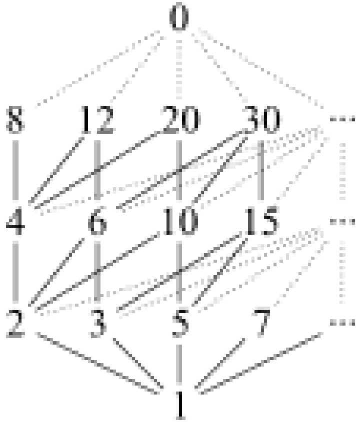
$$A = (x+2)(x-3)(x+3) \ ,$$
$$B = (x+2)(x+3)(x-5) \ .$$

It can be seen that the common divisors of A and B are: $(x+2), (x+3), (x+2)(x+3)$, and $\gcd(A, B) = (x+2)(x+3)$.

This common divisor is **greatest** in terms of divisibility relation, which says $a < b \Leftrightarrow a|b$. Hence,

$$(x+2) < (x+2)(x+3) \text{ because } (x+2)|(x+2)(x+3) \ ,$$
$$(x+3) < (x+2)(x+3) \text{ because } (x+3)|(x+2)(x+3) \ .$$

Therefore $\gcd(A, B) = (x+2)(x+3)$.



The Hasse diagram of positive integers ordered by divisibility.

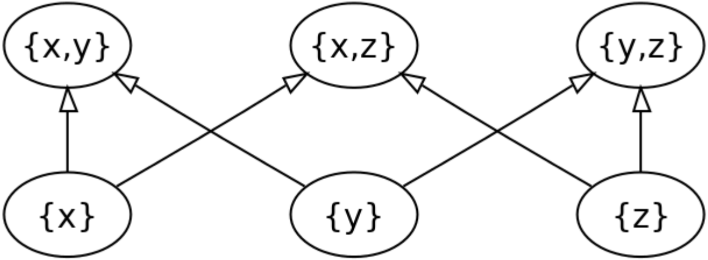
Let us show that $\subseteq$ establishes a partial order on any set A.

$$A \subseteq A \quad \text{(reflexivity)} \quad ,$$
$$A \subseteq B \wedge B \subseteq A \implies A = B \quad \text{(anti–symmetry)} \quad ,$$
$$A \subseteq B \wedge B \subseteq C \implies A \subseteq C \quad \text{(transitivity)} \quad .$$

Similarly, $\subset$ establishes a strict partial order on any set A.

$$\neg(A \subset A) \quad \text{(anti–reflexivity)} \quad ,$$
$$A \subset B \implies \neg(B \subset A) \quad \text{(asymmetry)} \quad ,$$
$$A \subset B \subset C \implies A \subset C \quad \text{(transitivity)} \quad .$$

$$A \subset B \subset C \implies (x \in A \implies x \in B \implies x \in C) \implies A \subset C \ .$$



The Hasse diagram of sets ordered by the inclusion relation.

Consider the powerset $\mathcal{P}(\{0,1\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$. Relation $\subset$ is a strict partial order on $\mathcal{P}(\{0,1\})$, i.e.

$$\emptyset \subset \{0\} \ , \quad \emptyset \subset \{1\} \ , \quad \{0\} \subset \{0,1\} \quad \{1\} \subset \{0,1\}$$

However, $\subset$ is not a strict total order, since

$$\{0\} \not\subset \{1\} \ , \quad \{1\} \not\subset \{0\} \ .$$

Elements $\{0\}$ and $\{1\}$ are not comparable by $\subset$, and hence the trichotomy property

$$\forall a, b \in X : a < b \vee b < a \vee a = b$$

does not hold.

A **total order** (a.k.a. **linear order** or a **chain**) is connex partial order. It is reflexive, anti–symmetric, transitive, and connex.

In other words, a **total order** is a partial order under which any two elements are comparable (connexity).

A **strict total order** is a trichotomous strict partial order.

In other words, a **strict total order** is a strict partial order under which any two elements are either comparable or equal (trichotomy).

A set with a total order on it is called a **totally ordered set**.

(strict) well order is a (strict) total order in which any non-empty subset has a least element.

A set with a well order relation on it is called a **well–ordered set**.

Definition 2 (Well–ordering principle)

Every non-empty set of non-negative integers contains a least element.

Corollary 1

The set of natural numbers $\mathbb{N}$ *is well–ordered.*

There are several kinds of extrema in a poset:

- Minimal / maximal element
- Least / greatest element
- Lower / upper bound
- Infimum / supremum

Let us take a look at the definitions of these elements, as well as on some examples.

Let P be a set partially ordered by R. Let $S \subseteq P$.

Element $m \in S$ is a **minimal** element of S if

$$\forall x \in S : xRm \implies x = m \ .$$

Minimal element is an element that is **not greater** than any other element in a set.

m is **maximal** element of S if

$$\forall x \in S : mRx \implies m = x \ .$$

Maximal element is an element that is **not smaller** than any other element in a set.

Let

$$S = \{\{d, o\}, \{d, o, g\}, \{g, o, a, d\}, \{o, a, f\}\}$$

be a set ordered by inclusion ($\subseteq$) relation.

A **minimal element** is an element that is **not greater** than any other element in the set.

A **maximal element** is an element that is **not less** than any other element in the set.

Considering the $\subseteq$ relation,

- a **minimal element** is an element that **is not a superset** of any other element in S.
- a **maximal element** is an element that **is not a subset** of any other element in S.

Let

$$S = \{\{d, o\}, \{d, o, g\}, \{g, o, a, d\}, \{o, a, f\}\}$$

be a set ordered by inclusion ($\subseteq$) relation.

Element $\{d, o\}$ is the **minimal element**, since the only element $s \in S$ such that $s \subseteq \{d, o\}$ is $\{d, o\}$ itself, it holds that

$$\{d, o\} \subseteq \{d, o\} \implies \{d, o\} = \{d, o\} \ .$$

Element $\{g, o, a, d\}$ is the **maximal element**, as the only element $s \in S$ such that $\{g, o, a, d\} \subseteq s$ is $\{g, o, a, d\}$ itself, it holds that

$$\{g, o, a, d\} \subseteq \{g, o, a, d\} \implies \{g, o, a, d\} = \{g, o, a, d\} \ .$$

Let

$$S = \{\{d, o\}, \{d, o, g\}, \{g, o, a, d\}, \{o, a, f\}\}$$

be a set ordered by inclusion ($\subseteq$) relation.

Element $\{d, o, g\}$ is **neither minimal nor maximal**, since

$$\{d, o\} \subseteq \{d, o, g\} \subseteq \{d, a, o, g\} \quad .$$

Element $\{o, a, f\}$ is both **minimal and maximal**, since relation $\subseteq$ contains only one pair of elements $(\{o, a, f\}, \{o, a, f\}) \in \subseteq$ that binds $\{o, a, f\}$ to itself, $\{o, a, f\}$ is the only subset and the only superset of iteself.

Hence, the minimal elements are $\{d, o\}$ and $\{o, a, f\}$ and the maximal elements are $\{g, o, a, d\}$ and $\{o, a, f\}$.

Minimal and maximal elements do not always exist.

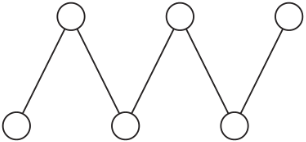
Let $S = [1, \infty) \subset \mathbb{R}$. Assume $m \in S$ is the maximal element. But there exists element $s = m + 1 \in S$ such that $m \leqslant s$ and $m \neq s$. Hence, there is a minimal element 1 and no maximal element in S.

Let $S = \{s \in \mathbb{Q} : 1 \leqslant s^2 \leqslant 2\}$ and recall that $s = \sqrt{2} \notin \mathbb{Q}$.

In general, $\leqslant$ is a partial order on some set S. If m is a maximal element of S and $s \in S$, there is a possibility that neither $m \leqslant s$ nor $s \leqslant m$. This leaves the possibility for the existence of many maximal elements.

There may be many maximal and minimal elements.

In a fense $a_1 < b_1 > a_2 < b_2 > \ldots a_n < b_n \ldots$ (see the image)



A fence consists of minimal and maximal elements only.

all a_i are minimal, and all b_i are maximal elements.

Let A be the set such that $|A| \geqslant 2$ and let

$$S = \{\{a\} : a \in A\} \subset \mathcal{P}(A) \ \ .$$

partially ordered by $\subset$.

The set S consists of singletons, which makes it a discrete poset – no two elements are comparable. For all $a', a'' \in A$ such that $a' \neq a$ it holds that

$$\{a'\} \cap \{a''\} = \emptyset \implies \{a'\} \not\subset \{a''\} \wedge \{a''\} \not\subset \{a'\} \ \ .$$

And therefore, every element $\{a\} \in S$ is **maximal** and **minimal**.

Let P be a set partially ordered by R. Let $S \subseteq P$.

Element $l \in S$ is the **least element** of S if

$$\forall x \in S : lRx \ .$$

The **least element** is an element that is smaller than or equal to any other element of S.

Element $g \in S$ is the **greatest element** of S if

$$\forall x \in S : xRg \ .$$

The **greatest element** is an element that is greater than or equal to any other element of S.

The notions of **maximal** and **minimal** elements are weaker than those of **greatest element** and **least element**.

Greatest and least elements are unique – a partially ordered set may have several maximal and minimal elements, but only one greatest and only one least element.

For totally ordered sets, the notions of maximal and greatest element coincide, and the notions of minimal and least elements coincide. They are called then **maximum** and **minimum**.

In fields like analysis, which deals with totally ordered sets only, the maximal, greatest, maximum are synonyms. The same holds for minimal, least, minimum.

A finite chain always has a greatest and a least element.

Proposition 1

The greatest element is a unique maximal element.

Proof.

Let P be a set partially ordered by R. Let $S \subseteq P$. Let g be the greatest element in S. We need to show that g is also a maximal element.

Since g is greatest element, it holds that $\forall s \in S : sRg$. We need to show that the implication $\forall s \in S : gRs \implies g = s$ holds. Indeed, it can be seen that by anti–symmetry of $\leqslant$, for all $s \in S$:

$$\forall s \in S : sRg \wedge gRs \implies g = s \ .$$

To show uniqueness, suppose there is another maximal element m such that gRm. By the definition of the greatest element, mRg, which implies that $m = g$. $\square$

Corollary 2

If there exists a greatest element, there is one maximal element, which is the greatest element itself.

Corollary 3

If there exists a least element, there is one minimal element, which is the least element itself.

The converse is not true: there can be several maximal elements with no greatest element.

Proposition 2

In a totally ordered set, the maximal element is the greatest element.

Proof.

Let P be a set partially ordered by R. Let $S \subseteq P$. Let m be the maximal element in S. By connexity, for all $s \in S$ either sRm or mRs. The condition sRm does not contradict with m being the greatest element in S. If mRs, by definition of a maximal element $mRs \implies m = s$. So we conclude that for all $s \in S : sRm$, and hence m is the greatest element in S. $\square$

Corollary 4

In a totally ordered set, the notions of the maximal element and the greatest element coincide, same as the notions of minimal element and a least element.

Proof.

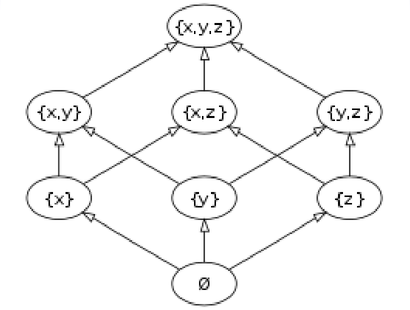
The proof is a direct consequence of propositions 1 and 2. $\square$

Let

$$S = \{\{d, o\}, \{d, o, g\}, \{g, o, a, d\}, \{o, a, f\}\}$$

be a set ordered by inclusion ($\subseteq$) relation.

There are no greatest nor least elements in S.



The powerset of $\{x, y, z\}$ ordered by $\subset$.

$\emptyset$ is the only minimal and least element of $\mathcal{P}(\{x, y, z\})$.

$\{x, y, z\}$ is the only maximal and the greatest element of $\mathcal{P}(\{x, y, z\})$.

The least and greatest element of a partially ordered set play a special role and are also called **bottom** and **top**, **zero (0)** and **unit (1)**, $\top$ and $\perp$.

If both exist, a poset is called a **bounded poset**.

In set theory, a set is **finite** iff every non-empty family of subsets has a minimal element when ordered by the inclusion ($\subseteq$) relation.

Let P be a set partially ordered by R. Let $S \subseteq P$.

Element $\lambda \in P$ is an **upper bound** of S if

$$\forall x \in S : xR\lambda \quad .$$

An **upper bound** of a subset S of a poset P is an element $\lambda \in P$ that is greater than or equal to every other element of S.

Corollary 5

The greatest element of $S \subset P$ (if it exists) is an upper bound of S.

Corollary 6

There may be many upper bounds.

Let P be a set partially ordered by R. Let $S \subseteq P$.

Element $\lambda \in P$ is a **lower bound** of S if

$$\forall x \in S : \lambda R x \ .$$

A **lower bound** of a subset S of a poset P is an element $\lambda \in P$ that is less than or equal to every other element of S.

Corollary 7

The least element of $S \subset P$ (if it exists) is a lower bound of S.

Corollary 8

There may be many lower bounds.

A poset with an upper bound is said to be **bounded from above** by that bound.

A poset with a lower bound is said to be **bounded from below** by that bound.

A poset is **bounded** if it has upper or lower bounds.

On the contrary, a poset without any bounds is called **unbounded**.

I.e., 5 is the lower bound of the set $\{5, 8, 42, 34, 13934\} \subset \mathbb{N}$. So is 4, so is 1, but 6 is not.

The set $\{42\} \subset \mathbb{R}$ is both an upper and lower bound, all other real numbers are either an upper bound or a lower bound of that set.

Every subset of $\mathbb{N}$ has a lower bound, since every such subset has a least element by the well–ordering principle.

An infinite subset of $\mathbb{N}$ cannot be bounded from above.

A proper infinite subset of $\mathbb{Z}$ can be bounded from below, or from above, but not from both sides.

An infinite subset of $\mathbb{Q}$ may or may not be bounded from below, and may or may not be bounded from above.

Every finite subset of a non–empty totally ordered set has both upper and lower bounds.

Let P be a set partially ordered by R. Let $S \subseteq P$.

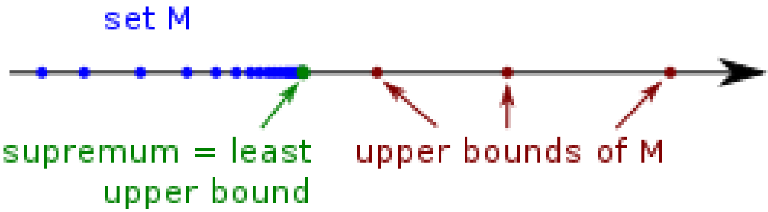
An upper bound I of S is a **supremum** ($\sup S$) if I is the least upper bound.

$$\forall U \in P : IRU \ .$$

A lower bound I of S is an **infimum** ($\inf S$) if I is the greatest lower bound.

$$\forall U \in P : URI \ .$$

An **infimum** is the greatest lower bound.
A **supremum** is the least upper bound.


set M
supremum = least upper bound
upper bounds of M

If a poset has the greatest element, it is one of its upper bounds. If a poset has upper bounds, the greatest element may not exist.

Consider the set of negative real numbers $\mathbb{R}^- = [-\infty, 0) \subset \mathbb{R}$. Any $x \in [0, \infty)$ is an upper bound of $\mathbb{R}^-$, but thre is no greatest element. Element 0 is a supremum (least upper bound) of $\mathbb{R}^-$, but the existence of a supremum also does not imply the existence of the greatest element.

Similarly, if a poset has the least element, it is one of its lower bounds, but the existence of an infimum or lower bounds does not imply the existence of a least element.

If $\sup S$ exists, it is unique (by the uniqueness of the least element).

If S contains the greatest element g, then $g = \sup S$, otherwise $\sup S \notin S$ or does not exist.

The existence of $supS$ may fail if the set has no upper bounds, or the set of upper bounds does not have the least element.

If $\inf S$ exists, it is unique (by the uniqueness of the greatest element).

If S contains the least element l, then $l = \inf S$, otherwise $\inf S \notin S$ or does not exist.

Existence of $\inf S$ may fail if the set has no lower bounds, or the set of lower bounds does not have the greatest element.

Finally, the concepts of minimal and least w.r.t. bounds also holds. There can be many minimal upper bounds with no least upper bound (supremum). Similarly, there may be many maximal lower bounds with no greatest lower bound (infimum).

The distinction between "minimal" and "least" is only possible when the given order is not a total order. In totally ordered sets, these terms mean exactly the same thing.

THANK YOU
FOR
YOUR
ATTENTION
ANY QUESTIONS?